Wave propagation analysis in laminated composite plates with transverse cracks using the wavelet spectral finite element method

Dulip Samaratunga a, b, Ratneshwar Jha a, b, S. Gopalakrishnan b

Department of Aerospace Engineering, Indian Institute of Science, Bangalore 560012, India
Raspet Flight Research Laboratory, Mississippi State University, 114 Airport Road, Starkville, MS 39759, USA

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A B S T R A C T

This paper presents a newly developed wavelet spectral finite element (WSFE) model to analyze wave propagation in anisotropic composite laminate with a transverse surface crack penetrating part-through the thickness. The WSFE formulation of the composite laminate, which is based on the first-order shear deformation theory, produces accurate and computationally efficient results for high frequency wave motion. Transverse crack is modeled in wavenumber–frequency domain by introducing bending flexibility of the plate along crack edge. Results for tone burst and impulse excitations show excellent agreement with conventional finite element analysis in Abaqus®. Problems with multiple cracks are modeled by assembling a number of spectral elements with cracks in frequency–wavenumber domain. Results show partial reflection of the excited wave due to crack at time instances consistent with crack locations.

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1. Introduction

Composite structures are increasingly used in aerospace, automotive, and many other industries due to several advantages over traditional metals such as superior specific strength and stiffness, lower weight, corrosion resistance, and improved fatigue life [1]. Of the main concerns that restrict wider usage of composites is the damage mechanisms which are not well understood. Most common damage types are delamination between plies, fiber–matrix debonding, fiber breakage, and matrix cracking resulting from fatigue, manufacturing defects, foreign object impact, etc. Currently available non-destructive evaluation (NDE) methods such as ultrasonic, radiography, and eddy-current are time consuming, expensive, and may need structural disassembly for inspection. In addition, these techniques are not suitable for damage detection during operational conditions such as aircraft or space structures in service. Therefore, a structural health monitoring (SHM) system capable of performing diagnostics online and efficiently is urgently needed. In recent years, Lamb wave based SHM techniques have been widely investigated for damage detection in composite structures [2–10].

SHM is essentially an inverse problem where the measured data are used to predict the condition of the structure. Therefore, to be able to differentiate between damage and structural features, prior information is required about the structure in its undamaged state. This is typically in the form of a baseline data obtained from the “healthy state” to use as reference for comparison with the test case. Alternatively, a mathematical model, such as a conventional finite element (FE) simulation, may be used to predict the structural response for comparison purposes. However, small damages such as transverse cracks (resulting from fiber breakage in composite structures) need high frequency Lamb waves for detection. Lowe et al. [11] observed a sinusoidal variation of the reflection coefficient with the ratio of notch width to A0 mode wavelength and determined that the reflection coefficient is maximized for small notch depths when the ratio stands at 0.5. In another observation, the reflection coefficient is maximized for the notch depth to A0 mode wavelength ratio of 0.3–0.4. These observations suggest that the diagnostic signal wavelength should be comparable to damage size for SHM. Therefore, high frequency excitation signals are generally used in SHM to identify minute damages present in structures. However, for high frequency excitation FE mesh size should be sufficiently refined (usually at least 20 nodes per wavelength) in order to reach acceptable solution accuracy which leads to large system size and computational cost. Thus FE simulation becomes prohibitive for SHM, especially onboard diagnostics with limited resources. The spectral finite element method (SFEM) yields models that are many orders smaller than FE, making it highly suitable for wave propagation based SHM. The transverse crack model developed here would be useful for understanding interactions between Lamb waves and

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cracks, optimizing excitation (interrogation) signal for damage
detection, and inference of damage type (based on damage-wave
interaction features).

The SFEM is essentially a finite element method formulated in
the frequency domain, which uses the exact solution to the wave
equations as its interpolating function. The frequency domain
formulation of SFEM enables direct relationship between output
and input through the system transfer function (or, frequency
response function). In SFEM, nodal displacements are related to
nodal tractions through frequency-wave number dependent stiff-
ness matrix and mass distribution is captured exactly to derive the
accurate elemental dynamic stiffness matrix. Consequently, in
the absence of any discontinuities, one element is sufficient to model
a plate structure of any length. Fast Fourier transform based spectral
finite element (FSFE) method was popularized by Doyle [12], who
formulated FSFE models for isotropic 1-D and 2-D waveguides
including elementary rod, Euler Bernoulli beam, and thin plates.
Gopalakrishnan et al. [13] extensively investigated FSFE models
for beams and plates—with anisotropic and inhomogeneous material
properties. The FSFE method is very efficient for wave motion
analysis and it is suitable for solving inverse problems. However,
the required assumption of periodicity in time approximation results in “wraparound” problem for smaller time window, which
totally distorts the response. In addition, for 2-D problems, FSFE
are essentially semi-infinite, that is, they are bounded only in one
direction [12]. Thus, the effect of lateral boundaries cannot be
captured and this can again be attributed to the global basis
functions of the Fourier series approximation of the spatial
dimension. The 2-D wavelet transform based spectral finite ele-
ment (WSFE) method presented by Mitra and Gopalakrishnan
[14–16] overcomes the “wraparound” problem and can accurately
model 2-D plate structures of finite dimensions.

Considerable work has been done for FE modeling of damages
in composite plates [17,18]; however, only a few researchers have
approached the problem using the spectral finite element method.
Palacz et al. [19] presented a spectral element to analyze trans-
verse crack in isotropic rod. Gopalakrishnan and associates have
formulated FSFE models for (open and non-propagating) trans-
verse cracks and delaminations in composite beams and plates
[20,21]. Levy and Rice [22] have shown that the slope disconti-
nuity at both sides of a crack due to the bending moments is propor-
tional to the bending compliance of the crack and nominal
bending stress. Using the expressions given in [22], Khadem and
Rezaee [23] obtained slope discontinuity at both sides of the
hypothetical boundary along the crack in terms of the character-
istics of the crack. Krawczuk et al. [9] used these expressions to
model a transverse crack using Fourier based spectral element for
isotropic plates. In a similar approach, Chakraborty et al. [24]
modeled wave propagation in a shear deformable composite
laminate with transverse crack using Fourier based spectral finite
element method. However, FSFE technique suffer from the ‘wrap-
around’ problem and cannot model finite structures, as noted
earlier. This study develops WSFE model for a non-propagating
and open transverse crack in shear deformable laminated composite
plates. The developed model has the ability to simulate wave
propagation in 2D finite structures with transverse cracks, without
the ‘wraparound’ problem of the FSFE method. In this procedure,
the plate with one crack is divided into two sub-elements, one on
either side of the crack. The placement of crack is parallel to one
of the major axes (X or Y). Following the concept of crack flexibility
[22,23], a discontinuity is introduced at the crack interface on the
angular displacements about the axis parallel to the crack. The
crack flexibility depends on crack parameters: length, depth at the
center, and shape of the crack. A continuity condition is imposed
at crack interface for all other angular and linear displacements, as
followed by other researchers [9,24]. Composite laminates with
multiple cracks are modeled similarly and the WSFE results are
validated with conventional FE analysis using Abaqus®.

The rest of paper is organized as follows. In the next section,
the WSFE modeling of shear deformable composite laminates is
introduced briefly. Next, the transverse crack formulation using
WSFE is presented. The accuracy and efficiency of the WSFE
modeling of crack is demonstrated by comparisons with FSFE
and FE simulations. Efficient simulation of multiple cracks using
WSFE is discussed. Concluding remarks are given in Section 5.

2. Wavelet spectral finite element modeling of shear
defeormable composite laminates

The 2-D WSFE presented by Gopalakrishnan and Mitra [16]
overcomes the “wraparound” problem present in FSFE and can
accurately model plate structures of finite dimensions. This is
due to the use of compactly supported Daubechies scaling functions [25] as basis for approximation of the time and spatial
dimensions. However, the WSFE plate formulation in [16] is based
on the classical laminated plate theory (CLPT) [26]. The CLPT based
formulations exclude transverse shear deformation and rotary
inertia resulting in significant errors for wave motion analysis at
high frequencies, especially for composite laminates which have
relatively low transverse shear modulus [27,28]. Wave propaga-
tion in composite laminates based on the first order shear
deformation theory (FSDT) [26], which accounts for transverse
shear and rotary inertia, yields accurate results comparable with
3-D elasticity solutions and experiments even at high frequencies
[27,28]. For isotropic materials FSDT is known to be exceptionally
accurate down to wavelengths comparable with the plate thick-
ness h, whereas CLPT is of acceptable accuracy only for wave-
lengths greater than, say, 20 h [29]. FSDT based 2-D WSFE model
for pristine composite laminates developed by the present authors is reported in [30,31]. Here, extensive time and frequency domain results are presented to show the significance of FSDT based spectral element formulation. The key steps in 2D WSFE element formulation for shear deformable composite laminates are presented below and the reader is referred to [30] for further details.

2.1. Governing differential equations for wave propagation

Consider a laminated composite plate of thickness $h$ with the origin of the global coordinate system at the mid-plane of the plate and $Z$ axis being normal to the mid-plane as shown in Fig. 1(a). Fig. 1(b) shows the corresponding nodal representation with degrees of freedom (DOFs). Using FSDT, the governing partial differential equations (PDEs) for wave propagation have five degrees of freedom: $u$, $v$, $w$, $\phi$, and, $\psi$. The terms $u(x, y, t)$ and $v(x, y, t)$ are mid-plane ($z = 0$) displacements along $X$ and $Y$ axes; $w(x, y, t)$ is transverse displacement in $Z$ direction, and $\phi(x, y, t)$ and $\psi(x, y, t)$ are the rotational displacements about $X$ and $Y$ axes, respectively. The displacements $w$, $\phi$, and $\psi$ do not change along the thickness ($Z$ direction). The quantities ($N_{xx}$, $N_{yy}$, $N_{xy}$) are in-plane force resultants, ($M_{xx}$, $M_{yy}$, $M_{xy}$) are moment resultants, and ($Q_n$, $Q_\phi$) denote the transverse force resultants.

There are five governing equations of motion for wave propagation as reported in the authors’ previous work [30], but only one EOM is presented here for conciseness. The first EOM based on the FSDT displacement EOM is presented here for conciseness. The orthogonal property of Daubechies scaling function approximation as reported in the authors’ previous work [30,31], but only one EOM is presented here for conciseness. Here, extensive time and frequency domain results are presented to show the significance of FSDT based spectral element formulation. The key steps in 2D WSFE element formulation for shear deformable composite laminates are presented below and the reader is referred to [30] for further details.

The governing PDEs (Eq. (1)) and boundary conditions (Eq. (3)) have three variables ($x$, $y$, and $t$), and derivatives with respect to them, making it very complex to solve. Therefore, Daubechies compactly supported scaling functions [25] are used to approximate the time variable which reduces the set of equations to PDEs in $x$ and $y$ only. Compactly supported scaling functions have only a finite number of filter coefficients with non-zero values, which enables easy handling of finite geometries and imposition of boundary conditions. The use of Daubechies compactly supported wavelets to solve partial differential wave equations is explained in detail in [14].

Let the time-space variable $u(x, y, t)$ be discretized at $n$ points in the time window $(0, t_f)$ and $t = 0, 1, ..., n-1$ be the sampling points, then $t = \Delta t$ where $\Delta t$ is the time interval between two sampling points. The function $u(x, y, t)$ can be approximated at an arbitrary scale as

$$u(x, y, t) = u(x, y, t_k) = \sum k u_k(x, y) \phi_k(t - k), \quad k \in \mathbb{Z}$$ (4)

where $u_k(x, y)$ are the approximation coefficients at a certain spatial dimension ($x$ and $y$) and $\phi_k(t)$ are scaling functions associated with Daubechies wavelets. The other translational and rotational displacements $v(x, y, t), w(x, y, t), \phi(x, y, t)$ and $\psi(x, y, t)$ are transformed similarly. The next step is to substitute these approximations into Eq. (1). The approximated equation is then multiplied with translations of the scaling function ($\phi_k(t - j)$, for $j = 0, 1, ..., n-1$) and inner product is taken on both sides of the equation. The orthogonal property of Daubechies scaling function results in the cancelation of all the terms except when $j = k$ and yields $n$ simultaneous equations.

$\sum_{k} A_{ij} \phi_k(t - j) = \sum_{j} A_{ij} \phi_k(t - j) + (A_{00} + A_{12}) \frac{\partial^2 v_j}{\partial x^2} + B_{11} \frac{\partial^2 \phi_j}{\partial x^2}$

$+ B_{12} \frac{\partial^2 \psi_j}{\partial x \partial y} + B_{22} \frac{\partial^2 \psi_j}{\partial y^2} + (B_{12} + B_{22}) \frac{\partial^2 \omega_j}{\partial y^2}$

$+ B_{20} \frac{\partial^2 \omega_j}{\partial y^2} \quad \text{for} \quad j = 1, 2, ..., n$ (5)

where $\Gamma^1$ is the first-order connection coefficient matrix obtained after using the wavelet extrapolation technique for non-periodic extension. Connection coefficients are the inner product between the scaling functions and its derivatives [32]. The wavelet extrapolation approach of Williams and Amaratunga [33] is used for the treatment of finite length data before computing the connection coefficients. This method uses polynomials to extrapolate the coefficients lying outside the finite domain and it is particularly suitable for approximation in time and the ease to impose initial values. This extrapolation technique addresses one of the serious problems with Fourier based SFE method, namely, the ‘wraparound’ problem caused by treating the boundaries as periodic extensions. By solving the ‘wraparound’ problem, WSFE method is able to handle short waveguides and smaller time windows efficiently. Next, the coupled PDEs are decoupled using eigenvalue analysis of Eq. (5). The decoupled form of the reduced PDEs given in Eq. (5) can be written as

$A_{11} \frac{\partial^2 \bar{u}_j}{\partial x^2} + (A_{00} + A_{12}) \frac{\partial^2 \bar{v}_j}{\partial x^2} + B_{11} \frac{\partial^2 \phi_j}{\partial x^2} + B_{12} \frac{\partial^2 \psi_j}{\partial y \partial x} + A_{02} \frac{\partial^2 \omega_j}{\partial y^2} + B_{22} \frac{\partial^2 \omega_j}{\partial y^2}$

$= -l_0 I_0 + I_1 \frac{\partial^2 \phi_j}{\partial x^2}$

where $\bar{u}_j$ is given by $\bar{u}_j = \Phi^{-1} u_j, \Phi$ is the eigenvector matrix of $\Gamma^1$ and $l_j (l = \sqrt{-1})$ are the corresponding eigenvalues. Following
2.3. Spatial (Y) approximation

The next step is to further reduce each of the transformed and decoupled PDEs given by Eq. (6) (and similarly for the other transformed governing equations and boundary conditions) to a set of coupled ordinary differential equations (ODEs) using Daubechies scaling function approximation in one of the spatial (Y) dimension. Similar to time approximation, the time transformed variable \( \tilde{u}_i \) is discretized at \( m \) points in the spatial window \((0, L_Y)\), where \( L_Y \) is the length in \( Y \) direction. Let \( \zeta = 0, 1, \ldots, m – 1 \) be the sampling points, then \( y = \Delta Y \zeta \) where \( \Delta Y \) is the spatial interval between two sampling points.

The function \( \tilde{u}_i \) can be approximated by scaling function \( \phi(\zeta) \) at an arbitrary scale as

\[
\tilde{u}_i(x, y) = \tilde{u}_i(x, \zeta) = \sum_l \tilde{u}_i(x) \phi(\zeta - l), \quad l \in \mathbb{Z}
\]  

where \( \tilde{u}_i \) are the approximation coefficients at a certain spatial dimension \( x \). The other four displacements \( \tilde{v}_i(x, y), \tilde{w}_i(x, y), \phi_i(x, y), \) and \( \psi_i(x, y) \) are similarly transformed. Following similar steps as the time approximation, substituting the above approximations in Eq. (6) and taking inner product on both sides with the translates of scaling functions \( \phi(\zeta - l) \), where \( i = 0, 1, \ldots, m – 1 \) and using their orthogonality property (which results in the cancelation of all the terms except when \( i = l \)), we get \( m \) simultaneous ODEs. Eq. (1), which is temporally reduced in Eq. (6), can be spatially reduced as

\[
A_{11} \frac{d^2 \tilde{u}_i}{dx^2} + (A_{00} + A_{12}) \sum_{l = 0}^{m - 1} \frac{\partial}{\partial \zeta} \phi(l) \tilde{u}_i + (B_{12} + B_{00}) \sum_{l = 0}^{m - 1} \frac{\partial}{\partial \zeta} \phi(l) \tilde{v}_i = -\delta_{ij} \tilde{u}_i - l_1 Y \phi_i
\]  

where \( N \) is the order of Daubechies wavelet and \( \Omega^1 \) and \( \Omega^2 \) are the connection coefficients for first- and second-order derivatives [32].

It can be seen from the ODE given by Eq. (8) that similar to time approximation, even here certain coefficients \( \tilde{u}_i \) near the vicinity of the boundaries \( (i = 0 \) and \( i = m – 1) \) lie outside the spatial window \((0, L_Y)\) defined by \( i = 0, 1, \ldots, m – 1 \). Here, after expressing the unknown coefficients lying outside the finite domain in terms of the unknown coefficients considering periodic extension, the ODEs given by Eq. (8) can be written as a matrix equation:

\[
A_{11} \left\{ \frac{d^2 \tilde{u}_i}{dx^2} \right\} + (A_{00} + A_{12}) \left\{ \frac{\partial \phi(l)}{\partial \zeta} \tilde{u}_i \right\} + B_{12} \left\{ \frac{\partial \phi(l)}{\partial \zeta} \tilde{v}_i \right\} + B_{00} \left\{ \frac{\partial \phi(l)}{\partial \zeta} \tilde{v}_i \right\} = -l_1 Y \phi_i
\]  

where \( \Delta Y^1 \) is the first-order connection coefficient matrix obtained after periodic extension.

The coupled ODEs given by Eq. (9) are decoupled using eigenvalue analysis similar to that done in temporal approximation. Let the eigenvalues be \( \beta_i \) (\( i = \sqrt{-1} \)), then the decoupled ODEs corresponding to Eq. (9) are given by

\[
A_{11} \frac{d^2 \tilde{u}_i}{dx^2} - i \beta_i (A_{00} + A_{12}) \frac{d\tilde{v}_i}{dy} + B_{12} \frac{d\phi_i}{dx^2} - i \beta_i (B_{12} + B_{00}) \frac{d\psi_i}{dy} = -l_1 Y \phi_i
\]  

where \( \beta_i \) (and similarly other transformed displacements) are given by \( \beta_i = \Psi^{-1} \tilde{u}_i; \Psi \) is the eigenvector and \( i \beta_i \) are the eigenvalues of connection coefficient matrix \( \Lambda^1 \). Exactly the same procedure is followed to obtain decoupled form for the other ODEs as presented in [30]. The natural boundary conditions (Eq. (3)) are also transformed (temporal and spatial approximations) similarly. The final decoupled ODEs are used for 2D WSFE formulation.

2.4. Spectral finite element formulation

The 2D WSFE has five degrees of freedom per node, \( \tilde{u}_i, \tilde{v}_i, \tilde{w}_i, \tilde{\phi}_i, \) and \( \tilde{\psi}_i \), as shown in Fig. 1(b). The decoupled ODEs presented above are solved for \( \tilde{u}_i, \tilde{v}_i, \tilde{w}_i, \tilde{\phi}_i, \) and \( \tilde{\psi}_i \) and the final displacements \( u(x, y, t), v(x, y, t), w(x, y, t), \phi(x, y, t) \) and \( \psi(x, y, t) \) are obtained using inverse wavelet transform twice for spatial \( Y \) dimension and time. For further steps, the subscripts \( j \) and \( l \) are dropped for simplified notations and all of the following equations are valid for \( j = 0, 1, \ldots, n – 1 \) and \( i = 0, 1, \ldots, m – 1 \) for each \( j \). Let us consider the displacement vector \( \tilde{d} \) as follows,

\[
\tilde{d} = \begin{pmatrix} \tilde{u}(x) \\ \tilde{v}(x) \\ \tilde{w}(x) \\ \tilde{\phi}(x) \\ \tilde{\psi}(x) \end{pmatrix}
\]  

Since the final set of equations (that is, Eq. (10) and the decoupled form of other ODEs) is ordinary equations with constant coefficients, we assume a solution of \( \tilde{d} \) as

\[
\tilde{d} = \tilde{d}_0 e^{-i\kappa x}, \quad \tilde{d}_0 = \begin{pmatrix} \tilde{u}_0(x) \\ \tilde{v}_0(x) \\ \tilde{w}_0(x) \\ \tilde{\phi}_0(x) \\ \tilde{\psi}_0(x) \end{pmatrix}
\]  

where \( \kappa \) stands for the wavenumber. Substituting Eq. (12) into the Eq. (10), the set of equations can be posed in Polynomial Eigenvalue Problem (PEP) form [13]. PEP uses the concept of the latent roots and right latent eigenvector of the system matrix (also called the wave matrix) for computing the wavenumber and the amplitude ratio matrix. The PEP of above Eq. (10) and other ODEs can be expressed as,

\[
\{A_2 \kappa^2 + A_1 \kappa + A_0 \} \{\tilde{d}_0\} = 0
\]  

The matrices \( A_2, A_1, A_0 \) are given in the Appendix.

The wavenumbers \( \kappa \) are obtained as eigenvalues of the PEP given by Eq. (13). Similarly, the vector \( \tilde{d}_0 \) is the eigenvector (representing wave amplitudes) corresponding to each of the wavenumbers. The solution of Eq. (13) gives a \( 5 \times 10 \) eigenvector matrix of the form

\[
\{R\} = \{d_01\}, \{d_02\}, \ldots, \{d_010\}
\]  

and the solution, \( \tilde{d} \) can be written as

\[
\{\tilde{d}\} = [R][\Theta]\{\alpha\}
\]  

where \( [\Theta] \) is a diagonal matrix with the diagonal terms

\[
[\Theta] = \{e^{-k_1 x}, e^{-k_2 x}, e^{-k_3 x}, e^{-k_4 x}, e^{-k_5 x}, e^{-k_6 x}, e^{-k_7 x}, e^{-k_8 x}, e^{-k_9 x}\}
\]
Here, \([\alpha]^T = \{C_1, C_2, \ldots, C_{10}\}\) are the unknown coefficients which can be determined as described in [30]. Finally, the transformed nodal forces \([\bar{F}^T]\) and transformed nodal displacements \([\bar{u}^T]\) are related by
\[
[\bar{F}^T] = [K^e] [\bar{u}^T]
\]
where \([K^e]\) is the exact elemental dynamic stiffness matrix. The solution of Eq. (16) and the assembly of the elemental stiffness matrices to obtain the global stiffness matrix are similar to the FE method. One major difference is that the time integration in FE uses a suitable finite difference scheme; however, the SFEM performs dynamic stiffness generation, assembly, and solution through a double do-loop over frequency and horizontal wavenumber. Although this procedure is computationally expensive, the problem size in SFEM is very small which keeps overall low computational cost. Another major difference is that unlike FE method, SFEM deals with only one dynamic stiffness matrix and hence matrix operation and storage require minimum computations.

3. Transverse crack modeling with wavelet spectral finite element

Although the WSFE method is applicable to general laminates, this study considers symmetric laminates wherein bending-extension coupling does not exist. For symmetric laminates, the equations of motion decouple into two sets of equations governing in-plane and transverse motions. The present model considers transverse motion of such laminates. A schematic diagram of the plate element with a crack (penetrating part-through the thickness and non-propagating) is shown in Fig. 2. The length of the element in the \(x\) direction is \(L_x\) and the plate has finite dimension of \(L_y\) in the \(y\) direction. The crack is located at a distance of \(L_t\) from the left edge of the plate (origin of the coordinate system) and has a length of \(2c\) in the \(y\) direction.

Two sets of nodal displacements \([\bar{w}_1, \bar{\phi}_1, \bar{\psi}_1]\) and \([\bar{w}_2, \bar{\phi}_2, \bar{\psi}_2]\) are assumed on the left and right hand side of a hypothetical boundary along the crack. In the wavenumber-frequency domain, these displacements are represented as \([\bar{w}_1, \bar{\phi}_1, \bar{\psi}_1]\) and \([\bar{w}_2, \bar{\phi}_2, \bar{\psi}_2]\) and expressed by
\[
\begin{bmatrix}
\bar{w}_1 \\
\bar{\phi}_1 \\
\bar{\psi}_1
\end{bmatrix} = [R]_{2\times 6} \begin{bmatrix} \Theta_1 \end{bmatrix}_{6 \times 1}, \quad \begin{bmatrix}
\bar{w}_2 \\
\bar{\phi}_2 \\
\bar{\psi}_2
\end{bmatrix} = [R]_{2\times 6} \begin{bmatrix} \Theta_2 \end{bmatrix}_{6 \times 1}
\]

where \([R]_{3\times 6}\) is the amplitude ratio matrix, and
\[
[\Theta_1]_{6 \times 6} = \text{diag}(e^{-ik_1x}, \ldots, e^{-ik_6x})
\]
\[
[\Theta_2]_{6 \times 6} = \text{diag}(e^{-ik_1(x + 2c)}, \ldots, e^{-ik_6(x + 2c)})
\]
are square matrices containing exponential terms. A total of 12 unknown constants of \([C_1]\) and \([C_2]\) are to be determined using the boundary conditions. These constants can be expressed as a function of the nodal spectral displacements using the boundary conditions as follows,

1. At the left edge of the element \((x = 0)\)
\[
\bar{w}_1 = \bar{q}_1, \quad \bar{\phi}_1 = \bar{q}_2, \quad \bar{\psi}_1 = \bar{q}_3
\]

2. At the crack interface \((x = L_t)\) for \([\bar{w}_1, \bar{\phi}_1, \bar{\psi}_1]\) and \(x = 0\) for \([\bar{w}_2, \bar{\phi}_2, \bar{\psi}_2]\)
\[
\bar{w}_1 = \bar{w}_2, \quad \bar{\psi}_1 = \bar{\psi}_2 \quad \text{(Continuity of transverse displacement and slope about X-axis across the crack)}.
\]
\[
\bar{\phi}_1 - \bar{\phi}_2 = \partial \bar{M}_{31x} \quad \text{(Discontinuity of slope about Y-axis across the crack)}.
\]
\[
\bar{M}_{31x} = M_{31x}, \quad \bar{M}_{31y} = M_{32y} \quad \text{(Continuity of bending moments)}.
\]
\[
\bar{M}_{321} = Q_{321} \quad \text{(Continuity of shear force)}.
\]

3. At the right edge of the element \((x = L - L_t)\) for \([\bar{w}_2]\)
\[
\bar{w}_2 = \bar{q}_4, \quad \bar{\phi}_2 = \bar{q}_5, \quad \bar{\psi}_2 = \bar{q}_6
\]

The terms \([q_1, \ldots, q_6]\) are nodal spectral displacements at the two ends. The term \(\partial\) is the non-dimensional bending flexibility of the plate along the crack edge. (The method to calculate \(\partial\) is given in Section 3.1) The quantities \(M_{31x}, M_{32y}\) and \(Q_{321}\) are the bending moments and shear force expressed in displacement terms, and \(M_{321}, M_{32y}, Q_{321}\) are obtained similarly. These boundary conditions can be expressed in the matrix form as
\[
[M] = \begin{bmatrix}
M_{11} & 0 & 0 \\
M_{21} & M_{22} & 0 \\
0 & M_{33} & 0
\end{bmatrix}, \quad [C] = \begin{bmatrix}
ut_1 \\
0 \\
0
\end{bmatrix}
\]

where \([u_1, \bar{q}_1, \bar{q}_3]^T\), \([u_2, \bar{q}_4, \bar{q}_6]^T\), \(M_{11}, M_{41} \in \mathbb{R}^{3 \times 6}\), \(M_{22}, M_{33} \in \mathbb{R}^{6 \times 6}\) and \(C = [C_1, C_2]^T\).

The shear force and bending moments at the left and right boundaries of the plate can be written as,
\[
\begin{pmatrix}
V_{x1}(0) \\
M_{31x}(0) \\
M_{31y}(0) \\
V_{x2}(L) \\
M_{32x}(L) \\
M_{32y}(L)
\end{pmatrix} = \begin{pmatrix}
\bar{Q}_1 \\
\bar{M}_{31x} \\
\bar{M}_{31y} \\
\bar{Q}_2 \\
\bar{M}_{32x} \\
\bar{M}_{32y}
\end{pmatrix} = \begin{pmatrix}
P_1 & 0 \\
0 & P_2
\end{pmatrix} \begin{pmatrix}
C_1 \\
P_1, P_2 \in \mathbb{C}^{3 \times 6}
\end{pmatrix}
\]

Sub-matrices \(M_i, i = 1, \ldots, 4\) and \(P_1, P_2\) can be found in the Appendix. Combining Eqs. (22) and (23),
\[
\begin{pmatrix}
\bar{Q}_1, \bar{M}_{31x}, \bar{M}_{31y}, \bar{Q}_2, \bar{M}_{32x}, \bar{M}_{32y}\end{pmatrix}^T = [K] \begin{pmatrix}
ut_1 \\
0
\end{pmatrix}
\]

where \([K]\) is the frequency dependent dynamic stiffness matrix of the spectral plate element with transverse (open and non-
propagating) crack and is given by
\[
[K] = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} [N_1 \ N_2], \quad \text{where} \quad N = [N_1 \ N_2 \ N_3 \ N_4] = [M]^{-1}
\] (25)
Here the stiffness matrix $K$ is a square matrix with dimensions of $6 \times 6$. The sub matrices $P_j$ ($j=1, 2$) and $N_i$ ($i=1, \ldots, 4$) have dimensions of $6 \times 12$ and $12 \times 3$ consecutively.

### 3.1. Bending flexibility due to a surface crack

The bending flexibility of the plate with a crack can be expressed in dimensionless form [9,22–24] as,
\[
\theta(y) = \left( \frac{H}{L_f} \right) a_{ba}^0 f(y) y, \quad y = \frac{y}{L_f}
\] (26)
where $H$ is the thickness of the plate, $F(y)$ is a correction function and $a_{ba}^0$ is the dimensionless bending compliance coefficient at the crack center given by
\[
a_{ba}^0 = \left( \frac{1}{H} \right) \int_0^{h_0} \xi(1.99 - 2.47\xi + 12.97\xi^2 - 23.117\xi^3 + 24.80\xi^4) dh, \quad \xi = \frac{h}{H}
\] (27)
for the range, $0 < \xi < 0.7$. Here $h$ is the depth of crack at any given $y$ and $h_0$ is the central crack depth. The crack shape function $f(y)$ is defined as,
\[
f(y) = e^{-2\sigma_0 y^2}, \quad \sigma_0 = \frac{C_0}{L_f}, \quad y_0 = \frac{y_0}{L_f}
\] (28)
where $\sigma$ is the base of natural logarithm. The correction factor $F(y)$ is given as,
\[
F(y) = \frac{2c/H + 3(1-\nu)\sigma_0 \int_0^{h_0} [1-f(y)]}{2c/H + 3(1-\nu)\sigma_0}
\] (29)

Fig. 3 presents the variation of $\theta$ for a few different values of crack parameters.

Now, in order to obtain the slope discontinuity ($\theta(y)M_{ax}$ in Eq. (20)) one would require the knowledge of bending moment at the crack location which is not known a priori. Therefore an initial assumption is made that the bending moment of the cracked plate is zero at the crack location. Using this, a first approximation of the displacement and stress field for the cracked plate can be obtained. This new result can be used again to replace $M_{ax}$, and this iteration can be continued until the convergence is achieved [24]. In this work, no iteration is performed as the objective is to show the qualitative change induced in the displacement field due to the presence of a crack.

### 4. Results and discussions

The formulated 2D WSFE is used to study transverse wave propagation in several composite laminates (graphite-epoxy, AS4/3501) having single and multiple part-through the thickness cracks. WSFE results are first compared with Fourier transform based spectral finite element results. Validations with (conventional) FE are performed using Abaqus® standard implicit simulations employing nine-node shell elements with reduced integration (S9R5), which is shear flexible and able to model multiple layers [34]. In the numerical examples provided, the properties of graphite-epoxy, AS4/3501 are taken as follows:
\[
E_1 = 144.48 \text{ GPa}, \quad E_2 = E_3 = 9.63 \text{ GPa}, \quad G_{23} = G_{13} = G_{12} = 4.128 \text{ GPa},
\]
\[
\nu_{23} = 0.3, \quad \nu_{13} = \nu_{12} = 0.02, \quad \text{and} \quad \rho = 1389 \text{ kg m}^{-3}.
\]
Time domain responses are studied using two types of input excitations, namely sinusoidal tone burst and impulse (Fig. 4).

Tone burst is essentially a windowed sinusoidal signal with a limited number of frequencies around the center frequency and it is narrow banded in the frequency domain. This type of input is used when a minimal number of frequencies are desired in the response to minimize wave dispersion. On the other hand, impulse excitation consists of multiple frequencies (due to the broadband nature in frequency domain) which cause multiple wave fronts being propagated through a waveguide. The spatial distribution of the input load along the $Y$-axis (in Fig. 2) is given by
\[
F(Y) = e^{-Y/\alpha^2}
\] (30)
where $\alpha$ (as specified for each analysis) is a constant and can be varied to change $Y$-distribution of the load. The 2D WSFE is formulated with Daubechies scaling function of order $N=22$ for temporal and $N=4$ for spatial approximations. The time sampling rate is $\Delta t = 1 - 2 \mu s$ (as specified for each analysis), while the spatial sampling rate $\Delta Y$ varies according to $\Delta Y = L_f / (m - 1)$ depending on $L_f$ and the number of spatial discretization points $m$.

#### 4.1. Comparisons with FSFE results

Wave propagation analysis of finite length structures using conventional spectral finite element analysis based on Fourier transform (FSFE) requires the structures to be damped or use of a throw-off element to dissipate energy out of the system [12,13]. In addition, the time window should be large enough to remove the wraparound problem associated with FSFE. The time window is dependent on the value of damping and the length of the structure, and it is required to be higher for lightly damped short length structures. WSFE is completely free from these constraints and the accuracy of solution is independent of such time window restrictions or energy dissipation by means of damping or throw-off elements [14]. Fig. 5 shows a composite plate (with transverse crack) considered for wave motion analysis using FSFE and the current WSFE formulation. The plate considered here is semi-ininitely long into $X$ direction and a crack is placed at $L_x = 0.5 \text{ m}$ distance away from the free end. Relatively large lateral spatial window ($L_y = 5 \text{ m}$) is chosen due to the Fourier series representation of the spatial dimension in FSFE where finite lateral boundary conditions cannot be imposed. A symmetric ply sequence of [0], with each ply having a thickness of 1 mm is considered. Time sampling rate is set at $\Delta t = 2 \mu s$ with the number of spatial
discretization points $m=64$ ($\Delta Y = 5/64$ m) and $\alpha = 0.1$ in Eq. (30). Fig. 6 shows transverse velocities at the tip of the plate due to a tone burst input load at the same location (edge AB in Fig. 5). The incident tone burst appears first in the response and the second wave packet with relatively smaller amplitude is the reflection from the crack. As mentioned earlier, the FSFE solution is highly dependent upon analysis time window as illustrated in Fig. 6. Three time windows of 1024 $\mu$s, 2048 $\mu$s, and 4096 $\mu$s are used for FSFE solutions, whereas the time window of 1024 $\mu$s is sufficient to obtain accurate WSFE results. It is observed that for $T_w=1024$ $\mu$s, FSFE solution is highly distorted and spurious wave components appear. The accuracy of FSFE results gradually increase with increasing time windows of 2048 $\mu$s (Fig. 6(b)) and 4096 $\mu$s (Fig. 6(c)). The FSFE prediction using the largest time window ($T_w=4096$ $\mu$s) is free from spurious distortions and matches WSFE results very well. Thus, the developed WSFE method yields substantial reduction in computational cost as the time window is directly related to the system size.

4.2. Validation with conventional finite element simulations

The developed 2D WSFE model for transverse crack is validated with FE simulations using Abaqus® standard implicit direct solver [34]. Fig. 7 shows the graphite-epoxy, AS4/3501, cantilever laminate used in this numerical example. The dimensions of the laminate are $L_x = L_y = 0.5$ m with a ply sequence of [0]$_{10}$ where each ply has a thickness of 1 mm. Responses are presented for the two types of inputs, tone burst and unit impulse (shown in Fig. 4). The tone burst has a central frequency of 20 kHz, while the unit impulse has duration of 50 $\mu$s and occurs between 100 and 150 $\mu$s, with a frequency content of 44 kHz. For WSFE analysis, time sampling rate is $\Delta t = 1$ $\mu$s with the number of spatial discretization points $m=64$ ($\Delta Y = 1/64$ m). The input load is applied along edge AB with the load variation given by Eq. (30) with $\alpha=0.05$.

The FE simulation in Abaqus® exploits symmetry of the problem about the centerline (parallel to edges AC and BD) and models half-plate. The model has 7688, 9-noded shell elements (S9R5) with the total number of nodes equal to 31,250 ensuring the model captures all the modes at the given excitation frequency. The crack is modeled using 62 line spring elements (LS6)
developed for modeling part-through cracks in shells [34]. The crack depth is 50% of the laminate thickness (that is, 0.005 m). The crack depth is 50% of the laminate thickness (that is, 0.005 m).

Fig. 8 shows the comparison of WSFE and FEM results at the tip and middle of the plate with the crack located in the middle of plate ($L_1 = 0.25$ m) and the tone burst input. It is observed that the formulated 2D WSFE element with transverse crack captures the wave scattering due to the crack and the results match very well with FE simulations. For the response at the plate tip (Fig. 8(a)), the incident wave (the first wave packet) propagates through the plate and part of this forward propagating wave is reflected due to the crack. This partial reflection is observed as the second wave packet with relatively smaller amplitude. The rest of the incident wave propagates beyond the crack and reflects completely from the fixed boundary (CD in Fig. 7), which is the third wave packet. The response at the center of the plate (Fig. 8(b)) also shows excellent match with FE including the fixed boundary reflection.

Fig. 9 shows the response of 2D WSFE with transverse crack has excellent match with FE simulation for the impulse input load. For the tip response (Fig. 9(a)), the incident impulse occurs between 100 and 150 $\mu$s, gets partially reflected from the crack around 400 $\mu$s, and the fixed boundary reflection starts to appear around 700 $\mu$s. The broad-band response at the center of the plate (Fig. 9(b)) also shows excellent match with FE including the fixed boundary reflection. Computational efficiency is one key advantage of WSFE over conventional FEM. For the results presented in Fig. 8, computational time for WSFE and FE (Abaqus®) simulations were 26 s and 4102 s, respectively (PC with 8-core, i7 processor). Thus, WSFE resulted in reduction of computational time by more than 2 orders of magnitude, even though only half-plate was modeled in Abaqus® using symmetry of the problem.

4.3. Identification of presence and location of crack using WSFE

The WSFE formulation presented in this paper may be used to identify presence and location of cracks through wave scattering due to a crack. The cantilever plate and loads used in the previous section are used for the numerical results given here. Fig. 10 shows...
the transverse velocities at the plate tip (which is also the excitation point) due to a tone burst input. For the healthy case (Fig. 10(a)), the first wave packet is the incident wave and the second larger wave packet is the reflection from the fixed boundary which arrives 582 μs after the excitation. Based on this arrival time, the group velocity of the flexural wave mode at this excitation frequency is calculated to be 1718 m/s. In the damaged plate response (Fig. 10(b)), there is an additional wave packet with smaller amplitude, which appears soon after the incident wave. Comparing this with the healthy plate response, it can be inferred that the additional wave packet is reflection from the crack. This reflected wave from the crack appears approximately 290 μs after the incident wave. With the group velocity data available (1718 m/s) the distance to the crack can now be determined to be (1718 × 290E – 06/2) = 0.25 m. Therefore the crack location can be determined when time of flight of reflected wave is used along with group velocity data. Fig. 11 shows the response of the plate for the same tone burst input with crack placed at 0.15 m and 0.35 m away from the tip (excitation point). In these responses too, the arrival time of additional wave packets indicate that they originate from the crack. In addition, when the crack is placed very close to the tip (Fig. 11(b)), there are multiple reflections occurring before the fixed boundary reflection arrives.

Fig. 12 shows the response for an impulse input load at the tip with crack placed at 0.15 m, 0.25 m, and 0.35 m from the excitation point. A consistent reflection of the input impulse from the crack is observed for all three locations of the crack. As indicated in the figure, the presence of crack does not change peak amplitude or the group velocity at which the pulse is propagated. However, there is an additional reflection from the crack arriving before reflection from the fixed boundary.

Fig. 8. Transverse velocity using 2D WSFE and FE (Abaqus®) for 20 kHz tone burst input applied at tip and response sensed at (a) tip (b) middle of the plate. Computational time for WSFE and FE simulations were 26 s and 4102 s, respectively, using a PC with 8-core, i7 processor.

Fig. 9. Transverse velocity using 2D WSFE and FE for impulse input applied at tip and response sensed at (a) tip and (b) middle of the plate.

Fig. 10. Response at the tip of 0.5 × 0.5 m² [0]ₙ laminate for a tone burst input: (a) healthy plate, and (b) crack at 0.25 m from the tip.
4.4. Multiple cracks analysis

The dynamic stiffness matrix of WSFE can be assembled in the frequency-wavenumber domain to study the presence of multiple cracks within a laminate. This assembling procedure is similar to the conventional FE method, except that the frequency domain approach is used [19,31]. A numerical example is presented here to demonstrate the ability of 2D WSFE method to simulate multiple cracks efficiently. Here a laminate with thickness and ply layup sequence different from above examples is used in order to show the flexibility of the model to analyze more general situations. An 8-ply graphite-epoxy, AS4/3501, laminate with a layup sequence of [0/90]_{2s} is considered. The laminate with two transverse cracks (Fig. 13(a)) is divided into two sub-laminates along a dashed line as shown in Figs. 13(b) and (c). The placement of this dashed line is somewhat arbitrary given that it is somewhere in between the two cracks. The in-plane dimensions are $L_X = L_Y = 0.5$ m and $L_Y = 1.0$ m while the thickness is kept at 0.008 m. The distance to crack 1 along the $X$-axis from the origin of its local coordinate system ($n_1$) is 0.375 m and the distance to crack 2 from $n_3$ is 0.25 m.

The dynamic stiffness matrix of each element (sub-laminate) is represented by Eq. (16). During assembly, nodes $n_2$ and $n_3$ coincide and the dynamic stiffness matrix for a laminate with two cracks is obtained. A tone burst load is applied at the tip of the laminate (at node $n_4$ in Fig. 13(c)) which has a spatial distribution given by Eq. (30) with $\alpha = 0.03$. Fixed boundary conditions are applied at the left edge of the laminate (node $n_1$).

Fig. 14 shows the transverse velocity response at the tip of laminate for three different cases: (a) healthy, (b) crack 1 present, and (c) both crack 1 and crack 2 present. For the healthy plate, the responses show the excitation wave packet (point $n_4$) and its reflection from the fixed boundary (point $n_1$). When crack 1 is

![Fig. 11. Response at the tip of 0.5 x 0.5 m [0]_{10} laminate for a tone burst input: (a) healthy plate, and (b) crack at 0.15 m from the tip, and (c) crack at 0.35 m from the tip.](image)

![Fig. 12. Response at the tip of 0.5 x 0.5 m [0]_{10} laminate for an impulse input: (a) healthy plate, (b) crack at 0.15 m, (c) crack at 0.25 m, and (d) crack at 0.35 m from the tip.](image)
introduced, there is an additional reflection arriving from the crack. When both cracks are present, distinct responses from the two cracks are clearly observed (Fig. 14(c)). Fig. 15 shows the transverse velocity response of each material point on the laminate along a straight line connecting nodes $n_4$ and $n_1$. The input tone burst excited at the tip starts propagating towards the left as...
5. Concluding remarks

A 2-D wavelet spectral finite element (WSFE) was presented for accurate modeling of high frequency wave propagation in anisotropic composite laminates with a transverse, non-propagating crack. The governing PDEs based on FSDT for laminated composite plates were used accounting for transverse shear strains and rotary inertia to produce accurate results for wave motion at high frequencies. Daubechies compactly supported, orthonormal scaling functions were employed for temporal and spatial approximations of the governing PDEs and the resulting equations were solved exactly to obtain the shape functions used in the element formulation. Since the developed element captured the exact inertial distribution, a single element was sufficient to model a laminate of any dimension in the absence of discontinuities. Further, due to the compact nature of Daubechies scaling functions, the WSFE was able to model finite waveguides without the signal ‘wraparound’ problem (associated with Fourier transform based spectral finite element). A transverse crack having an arbitrary length, depth, and located parallel to any one side of the plate was modeled in the wavenumber–frequency domain by introducing bending flexibility of the plate along the crack edge. The bending flexibility of the plate with crack was presented as a function of crack parameters, that is, crack length and depth at the center. Numerical examples using sinusoidal tone burst and broadband unit impulse excitations showed excellent agreement with conventional finite element simulations using shear flexible elements in Abaqus®. WSFE resulted in reduction of computational time by more than 2 orders of magnitude, even though only half-plate was modeled in Abaqus® using symmetry of the problem. WSFE modeling of transverse crack showed superiority over the FSFE results which showed signal distortions due to the “wrap around” problem for shorter time windows. To model multiple cracks within a laminate, the dynamic stiffness matrix of elements in the frequency-wavenumber domain were assembled, similar to the conventional finite element method. Time of flight estimations of waves reflected from cracks were successfully used to determine crack locations.

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Appendix

Matrices for polynomial eigenvalue problem (Eq. (13))

\[ A_2 = \begin{bmatrix}
-A_{11} & 0 & 0 & 0 & 0 \\
0 & -A_{66} & 0 & 0 & 0 \\
0 & 0 & -KA_{55} & 0 & 0 \\
0 & 0 & 0 & -D_{11} & 0 \\
0 & 0 & 0 & 0 & -D_{66}
\end{bmatrix} \]

\[ A_1 = \begin{bmatrix}
0 & -\beta(A_{12} + A_{66}) & 0 & 0 & 0 \\
-\beta(A_{12} + A_{66}) & 0 & 0 & 0 & 0 \\
0 & 0 & -iKA_{55} & 0 & 0 \\
0 & 0 & iKA_{55} & 0 & -\beta(D_{12} + D_{66}) \\
0 & 0 & 0 & -\beta(D_{12} + D_{66}) & 0
\end{bmatrix} \]

shown in Fig. 15(a). This wave packet interacts with crack 1 resulting in wave transmission and reflection due to the nature of the boundary created by the crack (Fig. 15(b)). The transmitted wave continues propagating towards the left, while the reflected wave heads towards the tip of the plate. The transmitted portion of the forward propagating wave is again scattered by crack 2 (Fig. 15(c)). The propagation direction of the incident wave packet is indicated by a solid arrow, while the reflected wave packet is indicated by a dashed arrow with a number to identify the crack causing the reflection. Following the same procedure, even more complex situations with multiple cracks may be simulated using WSFE, without much increase in computational time. Fig. 16 visualizes the entire laminate for the same case (as discussed in Fig. 15), showing transverse wave propagation and scattering due to the two cracks. Reflections from the lateral boundaries are observed in Fig. 16(c), which is a clear advantage of WSFE over FSFE.

Fig. 16. Full field view of flexural wave propagation in composite laminate with two cracks: (a) 190 μs, (b) 310 μs, and (c) 520 μs.
\[ \begin{bmatrix} -\beta^2 A_{66} + \gamma^2 I_0 & 0 & 0 \\ 0 & -\beta^2 A_{22} + \gamma^2 I_0 & 0 \\ 0 & 0 & -\beta^2 K A_{44} + \gamma^2 I_0 \\ 0 & 0 & -i/\kappa A_{44} \\ 0 & 0 & -i/\kappa A_{44} \\ 0 & 0 & -i/\kappa A_{44} \end{bmatrix} \]

Boundary condition matrices (Eq. (22)).

\[ M_1 = \begin{bmatrix} R_{11} & R_{12} & R_{13} & R_{14} e^{-i k_1 L} & R_{15} e^{-i k_1 L} & R_{16} e^{-i k_1 L} \\ R_{21} & R_{22} & R_{23} & R_{24} e^{-i k_1 L} & R_{25} e^{-i k_1 L} & R_{26} e^{-i k_1 L} \\ R_{31} & R_{32} & R_{33} & R_{34} e^{-i k_1 L} & R_{35} e^{-i k_1 L} & R_{36} e^{-i k_1 L} \end{bmatrix} \]

\[ M_4 = \begin{bmatrix} R_{11} e^{i k_1 L} & R_{12} e^{i k_1 L} & R_{13} e^{i k_1 L} & R_{14} & R_{15} & R_{16} \\ R_{21} e^{i k_1 L} & R_{22} e^{i k_1 L} & R_{23} e^{i k_1 L} & R_{24} & R_{25} & R_{26} \\ R_{31} e^{i k_1 L} & R_{32} e^{i k_1 L} & R_{33} e^{i k_1 L} & R_{34} & R_{35} & R_{36} \end{bmatrix} \]

References


