Estimate of the Disturbance Energy in an Asymmetric Vortex Wake

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Previous studies have suggested that the asymmetric vortex wake which develops behind an axisymmetric body at high angle of attack is the result of the amplification of a disturbance introduced at the tip of the body, implying that the asymmetric vortex wake problem is a stability problem. Measurements by previous researchers have shown the exponential growth of the so-called “distortion energy” in the wake, supporting the stability problem hypothesis. The definition of distortion energy, more generally described as a disturbance energy, chosen by these investigators requires a large set of measurements for a single result, and so another less experimentally-expensive definition of disturbance energy is desired. This study uses the potential flow solution for two vortices behind a circular cylinder in cross flow as a “virtual wind tunnel” to explore the validity of a particular definition of the disturbance energy. This definition, the product of the local section side force coefficient, the vortex separation distance, and the angle between the line connecting the vortices and the perpendicular to the incidence plane, is shown to exhibit the desired variations and is thus a plausible measure of the disturbance energy in the vortex wake.

Nomenclature

\( a \) = circular cylinder or local body cross-sectional radius
\( C_y \) = total body side force coefficient
\( D \) = body diameter
\( E \) = disturbance energy in asymmetric vortex wake
\( E_d \) = distortion energy in asymmetric vortex wake
\( U, U_0 \) = freestream or cross-flow velocity
\( U_a \) = velocity profile in asymmetric vortex wake
\( U_s \) = velocity profile in symmetric vortex wake
\( Z \) = axial coordinate along body
\( c_y \) = local section side force coefficient
\( d \) = vortex separation distance (dimensionless)
\( f_y \) = section side force (force per unit length)
\( t \) = time (dimensionless)
\( x \) = vortex location coordinate (dimensionless)
\( y \) = vortex location coordinate (dimensionless)
\( \alpha \) = angle of attack
\( \beta \) = vortex angle
\( \beta_0 \) = initial vortex angle
\( \gamma, \Gamma \) = vortex strength

I. Introduction

When an axisymmetric body is placed at sufficiently high angle of attack, the flow behind the body separates and forms a set of vortices behind the body which remain attached to the body. If the angle of attack is large enough, this vortex wake can become asymmetric, creating pressure distributions on the body that produce side forces and yawing moments, even though the body has a plane of symmetry with respect to the oncoming flow.

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This problem has been studied for a number of years. Two of the most commonly quoted survey papers on the subject, the reviews by Ericsson and Reding\textsuperscript{1} and by Hunt\textsuperscript{2}, are over 20 years old.

The reason for the extensive interest in this flow is its relation to the control problem of aircraft and missiles at high angles of attack. If the vortex wake behind the nose of an aircraft becomes asymmetric, the side force on the nose can generate a yawing moment which cannot be overcome by the aircraft’s rudder, particularly if the rudder’s effectiveness has been reduced because it is submerged in the wake of the aircraft wing. This problem has been studied extensively in flight tests on the X-31 as recently as 1994, and the report by Cobleigh\textsuperscript{3} describes this specific problem encountered during tests of the X-31 and subsequent attempts to eliminate or mitigate the unwanted yawing moments.

Although the asymmetric vortex wake presents a control problem, it also presents a control opportunity, in that if the side force on the nose of the aircraft could be controlled, then a mechanism would be available that would provide control in yaw when the aircraft is at high angle of attack and the rudder is ineffective because it is submerged in the wake of the aircraft’s main wing. Williams has produced a review\textsuperscript{4} of a number of forebody vortex control studies. Many of these focus on controlling the vortex wake through strakes, jets, or other passive and/or active mechanisms. One of the desirable characteristics of a control system, as pointed out by Williams and Bernhardt\textsuperscript{5}, is the ability to provide proportional control of the forebody asymmetry, or a degree of asymmetry (and thus side force and resulting yawing moment) that is proportional to the level of input from the system. The drawback of many of the control systems such as those reviewed by Williams\textsuperscript{5} is that rather than providing proportional control, they provide what could be considered a “bang-bang” control response, in which maximum system responses occur with no capability of producing intermediate levels of response.

In a previous study\textsuperscript{6}, the author presented a detailed discussion of the evidence for the asymmetric vortex wake as a stability-driven phenomenon, in that disturbances introduced at the tip of the body are amplified until the wake reaches a “saturated” state. At this point, typically one vortex breaks away from the body, and fine control of the wake asymmetry (and hence the side force) by disturbances introduced at the tip is lost, resulting in the “bang-bang” control characteristic just discussed. The argument is made that in order to understand the asymmetric vortex wake problem fully, the wake needs to be characterized in terms of a spatial and/or temporal stability problem, similar to the stability analysis of boundary layer flows. One quantity that is needed for such an analysis is an estimate of the “disturbance energy” contained in the asymmetric vortex wake. With the proper definition of the disturbance energy, it may be possible to perform a stability analysis to determine what parameters will affect the growth of disturbances in the vortex wake. Williams and Bernhardt\textsuperscript{7,8} have introduced one definition of the wake disturbance energy (referred to by those authors as the wake “distortion energy”), and their study will be discussed in some detail below. However, their definition of the disturbance energy is experimentally labor-intensive, requiring many measurements in the vortex wake to achieve a single result. Their definition also requires the deflection of the wake back to a symmetric state with a control input. The current study uses the potential flow for two point vortices behind a circular cylinder in a cross flow as a “virtual wind tunnel” to study a different definition of the wake disturbance energy. The advantages of this model are that the corresponding experiments would be simpler and less labor-intensive and no control input to the wake is required. The details of this model will be discussed, as will the results from the model and their similarities to the vortex flow

II. Initial Definition of Wake Disturbance Energy

As just pointed out, what is needed for a stability analysis of the vortex wake is a quantitative measure of the spatial growth rate of disturbances. A series of studies by Williams and Bernhardt provides some quantitative information along these lines.

In the first study, Williams and Bernhardt\textsuperscript{5} attempted to achieve proportional control of the vortex wake asymmetry through the unsteady bleed technique, alternating blowing and suction such that the net mass addition to the flow was zero. They tested a cone-cylinder model. The conical forebody had a 7.5-deg semiangle and a tip rounded to a radius of curvature of 0.476 cm. The overall slenderness ratio of the body was \( \frac{L}{D} = 10.7 \). Two ports for the unsteady bleed were drilled in the cone near the tip, one each on either side of the cone at an angle of 135 deg from the windward stagnation line. The bleed from each port could be activated independently of the other port. A circumferential ring of pressure taps was located just aft of the cone-cylinder intersection at \( x/D = 3.11 \). The pressures from these taps were integrated to obtain sectional side force coefficient at that location. The tests were conducted at a Reynolds number based on diameter of \( Re_D = 6,300 \) and at angles of attack of 45 deg and 55 deg. Results of these tests are shown in Fig. 1 (Fig. 4 in Ref. 5). At an angle of attack of 45 deg, the variation in sectional side force coefficient with respect to bleed coefficient was continuous (Fig. 1a), indicating that proportional control had been achieved. This control was also reversible, in that when the bleed was set to zero, the
side force coefficient would return to its original value. Williams and Bernhardt noted that after a certain value, an increase in the bleed coefficient magnitude would no longer produce a change in the magnitude of the side force coefficient, behavior consistent with a saturated system. The behavior of the wake supported the notion of the convective instability of the wake. However, at an angle of attack of 55 deg (Fig. 1b), the variation in $C_y$ was discontinuous, and no intermediate values of $C_y$ could be obtained in the vicinity of the discontinuity. This meant that proportional control had been lost and Williams and Bernhardt concluded that at this angle of attack, the vortex wake had attained a bistable nature, in which the vortices would lock into one of two bistable states and no control of the vortex wake was possible, other than causing a switch between these two states (a fuller discussion of the possibly bi-stable nature of the wake will be provided below). This is an example of the “bang bang” control response discussed earlier.

A quantitative measure of the development of wake asymmetry was provided by the next study by Bernhardt and Williams. The same model used in the previous experiment was used in this study. The wake was forced through the bleed ports as before. The model was tested at angles of attack of 45 deg and 55 deg over a Reynolds number range $6,300 < Re_D < 80,100$. In this set of experiments, however, the bleed from one port was increased at each condition until the side force coefficient $C_y$, obtained from the circumferential ring of ports at $x/D = 3.11$, was forced to be zero. The bleed from the other port was then increased until the wake was asymmetric, then the first port was once again adjusted until $C_y$ was zero. This procedure was repeated until a noticeable distortion of the wake velocity profile was observed. This procedure was performed in order to find the level of bleed required to make the wake symmetric without distorting the wake. The assumption was that in the absence of geometric disturbances on the model, the wake would be symmetric. Velocity profiles were measured along a direction perpendicular to the body-flow plane of symmetry for the symmetric wake with the bleed on and the corresponding asymmetric wake with the bleed off. A distortion energy $E_d$ was defined as

$$E_d = \int_{-l/2}^{l/2} \frac{U_a - U_s}{U_0^2} dy$$

where $U_a(y)$ was the mean velocity profile for the asymmetric wake, $U_s(y)$ was the mean velocity profile for the symmetric wake, and $l$ was the total length of the traverse. These measurements were repeated at a number of axial locations for $\alpha = 45$ deg and the resulting distribution $E_d(Z/D)$ is shown in Fig. 2. The distortion energy demonstrates exponential growth until approximately $Z/D = 4.5$, then the distortion energy levels off. Bernhardt and Williams state that the leveling off is consistent with a saturated end state. They also observed that this saturation occurred at approximately the point where one of the vortices separated from the body. A curve fit of the form $E_d = E_0 \exp(C Z/D)$ yielded a value of $C = 2.1$ for the spatial growth rate and $E_0 = 8.9 \times 10^{-5}$ as a measure of the initial disturbance. In addition to these results, Bernhardt and Williams also determined a “propagation speed” for disturbances in the vortex wake. A hot-wire anemometer was placed at a downstream location and the hot-wire signal and the pressure signal in the forcing system were measured simultaneously. The time delay between the pressure system input and the response of the wake as determined by the hot wire was measured for different axial locations of the hot wire. The slope of the delay time plotted against axial location was the propagation speed, and for an angle of attack of 45 deg and a Reynolds number of 6,300, the delay time increased linearly with downstream distance, yielding a propagation speed of $0.93U_\infty$ or 93% of the freestream speed. The propagation speed at an angle of attack of 55 deg was determined to be $0.76U_\infty$. Bernhardt and Williams noted that the system gain in terms of the ratio of thrust power output (as measured by the power presumably available in the side force, or $C_y$ $0.5pU_\infty D^2$) to the control power input ($pU_\infty^3d^2$) was on the order of $10^6$ at $Re_D = 30,000$.

III. Proposed Definition of Disturbance Energy

The disturbance energy defined by Bernhardt and Williams just discussed requires a large number of velocity measurements and is therefore rather time-consuming. The velocity must be surveyed along a line across the wake for the asymmetric wake, and then the appropriate amount of blowing must be introduced to force the wake to be symmetric, and then the velocity must be surveyed again. In addition to the number of measurements involved, the symmetric wake obtained by the forcing may or may not be identical to the symmetric wake that would develop in the absence of disturbances. Thus a new definition of the wake disturbance energy that is less labor-intensive to measure experimentally and does not require forcing to a symmetric state is deemed desirable.

Consider the two-dimensional vortex wake shown schematically in Fig. 3. This picture could represent the two-dimensional flow behind a circular cylinder or the cross-section perpendicular to the body axis of the asymmetric vortex wake of an axisymmetric body at angle of attack. Suppose a moment were exerted on the

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vortex pair to force the angle $\beta$ to zero by “twisting” the vortex wake. The moment exerted through the angle $\beta$ would represent an amount of work done on the wake. As a result of the wake asymmetry, a side force will be exerted on the body (cylinder or axisymmetric body at angle of attack). This force times the separation distance between the vortices would represent a couple. Therefore the product of the force, the separation distance, and the angle $\beta$ represents (at least in a dimensional sense) work done on the wake and is therefore a possible measure of the disturbance energy contained in the asymmetric vortex wake:

$$ E = f_Y d\beta $$

In this expression $E$ would actually be the energy per unit length, because $f_Y$ is the sectional side force acting on the cylinder or body. Without loss of generality, and in preparation for later computations, the disturbance energy is redefined in a dimensionless form as

$$ E = c_s d\beta $$

where now $c_s$ is the section side force coefficient and $d$ is the dimensionless vortex separation distance, made dimensionless by the cylinder or local body radius $a$.

### IV. Circular Cylinder Potential Flow as “Virtual Wind Tunnel”

To test the validity of the proposed wake disturbance energy given by Eqn. (2), the potential flow past a circular cylinder with two vortices behind the cylinder was used to “model” the flow past an axisymmetric body at angle of attack. The geometry of this flow is illustrated in Fig. 4 (Fig. 6.2 in Ref. 10). Föppl\textsuperscript{11} obtained a solution for steady, symmetric locations for the two vortices behind the body (to be discussed below). When the vortices are asymmetrically displaced from these positions, they will move away from the cylinder along asymmetric trajectories.\textsuperscript{12,13} The wake geometry and cylinder side force variations with time mimic the corresponding variations with axial position for the axisymmetric body at angle of attack, in both the initial development given by the potential flow solution and more generally by the unsteady, two-dimensional vortex-shedding phenomenon, which is the time analog to the long slender body at angle of attack from which multiple wake vortices may break away. As a result of these correspondences, this two-dimensional flow has been used numerous times in describing and computing the flow past the axisymmetric body at angle of attack (see e.g. the paper by Hall\textsuperscript{14}). It is not the goal of this paper to develop another detailed time-space analogy model to attempt to describe in detail the behavior of the asymmetric vortex wake, but rather to use the results of the potential flow model to determine if the proposed disturbance energy definition is reasonable. Comparisons and correlations between the 2-D and 3-D flows will be made as deemed appropriate, to support the validity of the definition.

The complex potential for the flow shown in Fig. 4 for two vortices of arbitrary strength and position is given (in terms of dimensional quantities) by\textsuperscript{11}

$$ W = U \left( z + \frac{a^2}{z} \right) + \frac{i \Gamma_1}{2\pi} \left( -\log(z - z_1) + \log \left( z - \frac{a^2}{z_1} \right) - \log z \right) $$

$$ + \frac{i \Gamma_2}{2\pi} \left( -\log(z - z_2) + \log \left( z - \frac{a^2}{z_2} \right) - \log z \right) $$

In this equation, $z$ is the position variable, $z_1$ and $z_2$ are the complex positions of the point vortices, and $z_1^*$ and $z_2^*$ are the complex conjugates of the vortex positions. The corresponding complex dimensional velocity is given by

$$ \frac{dW}{dz} = U \left(1 - \frac{a^2}{z^2} \right) + \frac{i \Gamma_1}{2\pi} \left( -\frac{1}{z - z_1} + \frac{1}{z - \frac{a^2}{z_1}} - \frac{1}{z} \right) + \frac{i \Gamma_2}{2\pi} \left( -\frac{1}{z - z_2} + \frac{1}{z - \frac{a^2}{z_2}} - \frac{1}{z} \right) $$

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At any given instant the velocity of one vortex can be computed from this equation by computing the velocity induced at the position of the vortex in question induced by the other vortex. This velocity can be used to compute the change in position of the vortex through a given time step, and so the vortex trajectories may be determined by integrating the velocity equation using a Runge-Kutta integration scheme. This is the approach taken in this study.

Föppl’s solution for steady vortex positions behind the cylinder, mentioned above, is given by

$$y = \frac{1}{2} \left( r - \frac{1}{r} \right), \quad x = \sqrt{r^2 - y^2}$$  \hspace{1cm} (5)

where $r$ is the radial distance from the center of the cylinder. These curves are shown in Fig. 5. The corresponding vortex strengths are given by

$$\Gamma = \pm 2y \left( 1 - \frac{1}{r^4} \right)$$  \hspace{1cm} (6)

where $y$ is obtained from Eqn. 5.

At any instant, the sectional force coefficients may be obtained from the vortex positions and strengths using the following formula, obtained by using the Blasius formula to integrate the complex velocity (see Ref. 10):

$$\frac{1}{4\pi} \left( C_x - iC_y \right) = \frac{i\gamma_1}{z_1^2} + \frac{i\gamma_2}{z_2^2} + \gamma_1 \frac{z_1^*}{z_1 z_1^* - 1} + \gamma_2 \frac{z_2^*}{z_2 z_2^* - 1} - (\gamma_1 \gamma_2 + \gamma_1^2) \frac{1}{z_1} - (\gamma_1 \gamma_2 + \gamma_2^2) \frac{1}{z_2} - \gamma_1 \gamma_2 \left( \frac{z_1^*}{1 - z_2 z_1^*} + \frac{z_2^*}{1 - z_1 z_2^*} \right)$$  \hspace{1cm} (7)

In this expression, $\gamma_1$ and $\gamma_2$ are the dimensionless vortex strengths, and all vortex positions have been made dimensionless by the cylinder radius $a$.

In the simulations to be described below, the initial vortex positions were chosen as a small asymmetric displacement from the symmetric Föppl solution at a chosen initial value of $r = r_0$. For the given value of $r_0$, the $y$ and $x$ coordinates were computed from Eqn. (5) and the corresponding vortex strengths from Eqn. (6). An initial (small) value for the angle $\beta$ shown in Fig. 3, defined as $\beta_0$, was then chosen and the vortices were displaced accordingly. The Runge-Kutta integration was then started, usually running to a dimensionless time $t' = Ut/a = 20$ over 100 steps. Previous experience with this program had shown that the solution trajectories were usually independent of the time step as long as the step was sufficiently small.

V. Results

Figure 6 shows the results for the vortex trajectories for $r_0 = 2$ and $\beta_0 = 0.5$ deg. The Föppl solution curves are also shown in the figure. These trajectories demonstrate the general tendencies of all the solutions, in particular that one vortex moves away from the cylinder while the other moves closer to it before moving away. Both vortices wind up on one side of the $x$-axis, which would correspond to an edge view of the incidence plane for an axisymmetric body at angle of attack.

Figure 7 shows the results for the development of the side force coefficient with dimensionless time. As the asymmetry initially develops, the side force gets larger. As the vortices move away from the body and their influence diminishes, the side force returns to zero. If this were a 2-D unsteady shedding problem, the side force coefficient would oscillate between positive and negative maxima. The corresponding axial development of sectional side force on an axisymmetric body at angle of attack with multiple breakaway vortices would illustrate a similar oscillation, with the peaks corresponding roughly to the breakaway positions of the vortices. Figure 7 could be thought of as the first oscillation in such a cycle. In a situation in which only the forebody flow is of interest, the initial development of the side force coefficient is the critical quantity.

Figure 8 shows the development of the wake geometry through the vortex angle $\beta$. The initial value of this angle, $\beta_0$, was 0.5 deg in this case. The angle gradually increases and then asymptotes to a maximum value as the vortex pair moves away from the body, essentially experiencing no more influence from the body and moving under the influence of only the velocity field induced by the vortex pair.
Figure 9 shows the development of the wake disturbance energy defined above. The quantity plotted is

\[
\ln \left( \frac{E}{E_0} \right) = \ln \left( \frac{c_s d \beta}{(c_s d \beta)_{\text{ref}}} \right);
\]

(8)

If this quantity demonstrates a linear variation with time, then the wake disturbance energy as so defined is growing exponentially in time, as did the wake distortion energy measured by Bernhardt and Williams\textsuperscript{7} and illustrated in Fig. 2. As can be seen in Fig. 9, the initial development of the wake disturbance energy does indeed show an exponential increase in time, indicating that the proposed disturbance energy definition does produce the proper behavior. The details of this variation will be addressed below.

In this model of the flow, there are only two parameters, the initial placement of the symmetric vortex pair, characterized by \( r_0 \), and the initial disturbance level, characterized by \( \beta_0 \). The effects of these two parameters were studied in some detail. Figure 10 shows the effect of \( \beta_0 \) on the development of the vortex wake trajectories. In this figure, one of the two initial angles was chosen as 10 deg, to exaggerate the effect of the initial condition. The standard values used for \( \beta_0 \) in the remainder of the studies were 0.5 deg, 1.0 deg, and 2.0 deg. Figure 10 shows that, after some initial variations, the vortices always settle down to one particular pair of trajectories. This was found to be true for all of the variations of initial disturbance angle. Particular trajectories were determined not by initial disturbance level \( \beta_0 \) but by initial position \( r_0 \), as will be discussed below.

Figures 11, 12, and 13 show the effects of the initial disturbance level on the development of the section side force coefficient, the vortex angle, and the disturbance energy, respectively, for values of \( \beta_0 \) of 0.5, 1.0, 1.5, 2.0, and 10 deg and a value of \( r_0 = 2 \) for all cases. The effect of the initial disturbance on the section side force coefficient is to cause it to reach a maximum in magnitude faster as the initial disturbance level goes up, although the maximum magnitude appears to be independent of the initial disturbance level. These variations are very similar to computational results achieved by Levy, Hesselink, and Degani\textsuperscript{15,16} in their study of the effects of initial disturbance levels on side force development on an axisymmetric body at angle of attack. The body was an ogive-cylinder with an ogive forebody of slenderness ratio \( L/D \) of 3.5 and a cylindrical afterbody of slenderness ratio \( L/D \) of 4. The computations utilized a time-marching thin-layer Navier-Stokes method and were conducted for flow conditions of \( M = 0.28, \alpha = 40 \text{ deg} \), and \( Re_D = 3*10^6 \). A bump, shown in Fig. 14 (Fig. 1 in Ref. 15) was added to the body geometry at a circumferential angle of 90 deg from the windward stagnation line. The bump height was varied between \( h/D = 0.0008 \) and \( h/D = 0.01 \). The results for the distribution of sectional side force coefficient \( C_Y \) are shown in Fig. 15 (Fig. 8 in Ref. 15). The similarities in Figs. 11 and 15 are rather intriguing. In Fig. 15, an increase in bump height causes a corresponding increase in the rate at which the maximum \( C_Y \) value is achieved, up until a certain point. The curves corresponding to the largest values of \( h/D \) essentially overlap, indicating that further increases in bump height are not producing corresponding increases in rate of side force development. The difference between Figs. 11 and 15 and the corresponding flows is that in Fig. 15, the side force always begins at or near zero, whereas in Fig. 11, the side force begins with a different non-zero value for each value of \( \beta_0 \) as a result of the initial asymmetry of the wake. It might be expected that in Fig. 15 there would be a small, discontinuous jump in the side force coefficient when the flow encountered the bump, but such a jump is not visible in the plot. It should also be noted from this figure that the maximum \( C_Y \) produced is the same for all bumps, another similarity with the circular cylinder flow of Fig. 11. Levy, Hesselink, and Degani noted that the maxima and zero crossings in the side force distribution corresponded to vortex separations, observing that “there is an excellent correlation between the location where the vortices curve away from the body and the location of the maxima, minima, and zero crossing of the sectional side-force coefficient” (Ref. 15, p. 5). The existence of a “global” maximum value of the side force coefficient which occurs at the point where a vortex separates from the body is further evidence supporting the notion that vortex separation occurs as the result of the saturation of the vortex wake. Levy et al. note that the oscillating side force distribution indicates multiple separations of vortices as the flow develops on the cylindrical afterbody, as discussed above.

Figure 12 shows the effects of the initial disturbance on the development of the vortex wake angle. These results are similar to those of Fig. 11, in that an increase in initial disturbance starts the flow further down the path to the final asymmetric state, although the paths to asymmetry seem to be the same. Figure 13 is rather intriguing. It shows that the initial growth of the wake disturbance energy is the same regardless of the initial disturbance, but that the maximum energy level reached decreases, and the time at which the peak energy level is reached also decreases, as the initial disturbance increases. Of course, the denominator in the quantity given by Eqn. (8) and plotted in Fig. 13 is smaller for the smaller initial disturbance levels, so in reality all of the curves are the same. In fact, by adding constants to the abscissa and ordinate values in Fig. 13, it is possible to shift all curves onto a single curve.
Figure 16 does illustrate the significant effect of initial location parameter $r_0$ on the vortex trajectories, for initial disturbance $\beta_0 = 0.5$ deg. The values of $r_0$ used were 1.5, 2.0, and 2.5. Recall that the corresponding values of $\Gamma$ were obtained from Eqn. (6). As the initial position moves further out from the cylinder, the initial vortex separation distance becomes larger, and the influence both of the vortices on each other and of the image vortex system inside the cylinder on the wake vortices appears to decrease and the motion of the vortices becomes less erratic (note that for $r_0 = 2.5$, the tendency for the vortex to move toward the cylinder before moving away is decreased).

Figure 17 shows the effects of initial vortex location parameter $r_0$ on the development of the side force coefficient. In each case, the side force coefficient first increases in magnitude and then decreases. It is interesting to note that the maximum magnitude of the side force coefficient first increases with increasing $r_0$ and then decreases. However, this effect is not seen in Fig. 18, which shows the effect of initial location on the wake disturbance energy development. In all cases, the disturbance energy first grows exponentially and then levels off, indicating a saturated state, before decreasing. As the initial location parameter $r_0$ increases, the maximum energy level decreases. The initial exponential growth of the disturbance energy and the effect of the flow separation point on the development of wake disturbance energy, similar to the straight line of the data of Bernhardt and Williams' shown in Fig. 2. The slope of each of the lines in Fig. 19 (the coefficient of “x” in each of the linear regression curvefit expressions) is the exponent describing the growth of the disturbance. It can be seen that increasing $r_0$ causes a decrease in the slope and hence the exponent of the growth. This is in line with results obtained for axisymmetric bodies at angle of attack. Bodies having a sharper tip tend to exhibit more unstable vortex wakes, a trend attributed by Keener and Chapman\(^\text{17}\) to vortex “crowding” near the tip and the subsequent increased influence of each vortex on the other. Keener and Chapman compared the onset of asymmetric wakes behind slender bodies and delta wings with large sweepback angles, where the separation point is fixed. The similarities between the two phenomena led them to conclude that the asymmetric wake is the result of an inviscid instability that occurs when the two wake vortices are “crowded together” near the tip of the body. One vortex moves away from the body and the other vortex moves underneath the first. A larger nose angle would increase the distance between the vortices and reduce the vortex “crowding,” so that a higher angle of attack would be required for the onset of wake asymmetry. The current results support this conjecture in that vortices that start off closer together have a higher rate of development of wake disturbance energy. It seems apparent that there should be some sort of relationship among the initial location parameter $r_0$, the angle of attack $\alpha$, and the axisymmetric body nose semi-angle $\theta_c$. Most likely, the relationship is between $r_0$ and the relative angle of attack parameter $\alpha/\theta_c$. However, the relationship is not obvious (at least to this author) and will require further consideration before it is established.

The description of the vortex wake behavior by Keener and Chapman\(^\text{17}\) just discussed is connected with the results of Levy, Hesselink and Degani\(^\text{15,16}\) discussed above in the following aspect: at the point where one vortex breaks away and the second vortex moves underneath it, as described by Keener and Chapman, Levy et al. noted that the side force coefficient reached a maximum. The evidence of Bernhardt and Williams’ suggests that their distortion energy reaches a maximum (i.e. a saturated state) at the point where the first vortex breaks away from the body. Bridges\(^\text{15}\) and Bridges and Hornung\(^\text{19}\) observed similar behavior, noting that the loss of ability to control the wake asymmetry (in their case through a disturbance introduced by an elliptic cross-section tip) corresponded to the point where one vortex broke away from the body and the second moved underneath the first. In Fig. 16, the vortex locations marked with the “+” symbol indicate the vortex positions in each pair at the instant where the wake disturbance energy reaches a maximum in Fig. 18 for each value of $r_0$. In Fig. 16, these positions correspond to the situation where the vortex closest to the cylinder is about to cross the $x$-axis, another example of the qualitative agreement between the circular cylinder model and the wake of the axisymmetric body at angle of attack.

In his analysis\(^\text{12}\) of the stability of the Föppl vortex wake, Smith assumed that the vortices would be displaced from positions on the Föppl curve $z_1$ and $z_2$, where $z_2 = z_1^*$ is the complex conjugate of $z_1$, to $z_1 + \alpha$ and $z_2$ to $z_2 + \beta^*$. Smith then formed equations for the variables

$$x_1 + iy_1 = \alpha + \beta, \quad x_2 + iy_2 = \alpha - \beta$$

(9)

The complex variable $x_1 + iy_1$ (which is not $z_1$) represents the symmetric portion of the assumed disturbance displacement, and the variable $x_2 + iy_2$ (which is not $z_2$) represents the antisymmetric portion of the assumed
disturbance displacement. Smith then formed linearized differential equations for the development of the quantities \(x_1+iy_1\) and \(x_2+iy_2\). Solutions were obtained of the form

\[
x_1 = h_1 e^{\lambda t}, \quad y_1 = k_1 e^{\lambda t}, \quad x_2 = h_2 e^{\mu t}, \quad y_2 = k_2 e^{\mu t}
\]

In Smith’s notation, the initial vortex positions were characterized by the radial distance from the origin \(r\) (corresponding to \(r_0\) above), which was the magnitude of the initial complex position variable. Smith found that the solutions yielded negative values for \(\lambda\) in Eqn. (10) for all values of \(r > 1\), which meant that symmetric disturbances were stable. However, one of the roots for \(\mu\) in Eqn. (10) was positive for all values of \(r > 1\), meaning that antisymmetric disturbances were unstable. His solution for \(\mu\) is given by

\[
\left(\frac{\mu}{U}\right)^2 = \frac{1}{r^{10}} \left(1 - 3r^2 + 3r^4 + 3r^6\right)
\]

The value of \(\mu\) is essentially the growth rate of an asymmetric disturbance.

If, as is often the case, the energy contained in a disturbance is proportional to the square of the amplitude of the disturbance, then the growth rate of the energy should be twice that of the growth rate of the amplitude. Figure 20 is a plot of twice the value of \(\mu\) as a function of \(r\) obtained from Eqn. (11). Also shown on this plot are the slopes obtained from Fig. 19. The agreement between the two curves is very good, although not exact. The value \(2\mu\) consistently exceeds the slopes from Fig. 19 by approximately 3%. However, the overall agreement, particularly the correct variation with the initial position \(r\), supports the proposition that the disturbance energy given by Eqn. (2) reasonably and accurately represents the behavior of the vortex wake.

VI. Methods of Measuring the Wake Disturbance Energy and Future Work

In order to implement the proposed definition of wake disturbance energy, the axial development of the sectional side force coefficient, the vortex separation distance, and the vortex angle must be measured for each value of the independent parameter being studied. The sectional side force coefficient may be determined by circumferential rings of pressure taps on the model. It was hoped that the side force coefficient would correlate with the pressures on the sides of the model, meaning that only two pressure taps would be required at each axial location. However, investigations using the circular cylinder potential flow model showed that the various quantities of interest did not exhibit the desired behaviors, necessitating the determination of the actual side force coefficient at each axial location. The vortex angle \(\beta\) and the vortex separation distance \(d\) may be determined by a visualization of the cross section of the wake at each of the axial locations of pressure tap rings. The apparatus to do this is shown in Fig. 21 (Fig. 6 in Ref. 18). Fluorescein dye is injected from within the model. A laser sheet is adjusted such that it strikes the model perpendicular to the model's axis. Optical equipment (in Fig. 6, a mirror and a video camera) is used to image the vortex wake. Representative images of the vortex wake so obtained are shown in Fig. 22 (Fig. 13 in Ref. 9). While the data acquisition required for the proposed wake disturbance energy definition is somewhat labor-intensive, it is believed to be less so than the definition proposed by Williams and Bernhardt. A complication arises from the fact that the flow visualization method just described is usually done best in water (although it can be done with smoke in a wind tunnel, but somewhat less effectively), whereas pressure measurements made from pressure taps on a model are usually accomplished more easily in air. In this case, it might become necessary to test the model in both a water tunnel facility and a wind tunnel facility, with care being taken to match Reynolds numbers and the axial locations of pressure measurements and wake cross section images. If a reliable method of performing both measurements quickly can be developed, then the effects on the vortex wake asymmetry of a wide range of parameters such as initial disturbance, relative angle of attack, etc. may be studied.

Figure 22 was chosen deliberately to represent the vortex wake images, because it also illustrates the correspondence between the vortex positions in the wake of an axisymmetric body at angle of attack (the positions scaled by the local radius of the body) and the Föppl solution for steady vortex positions behind the circular cylinder in cross flow. The measured vortex positions correspond to different orientations of the elliptic cross section tip. The fact that all of these vortex positions lie on or near the Föppl curve suggests that the scalings of the two vortex flows are very similar and validates the use of the Föppl solution as the initial condition for the vortex trajectory studies presented above.
As stated at the beginning, the objective of this study was to define a relatively-simple and easily-acquired measure of the disturbance energy contained in the asymmetric vortex wake. Usually, the purpose of a stability analysis is to determine what parameters affect the growth of disturbances in a particular flow, and that growth is often represented by an energy associated with the disturbance. The energy is an important quantity because of the breakdown in system behavior associated with the energy reaching a maximum level or the system reaching a saturated state. If the proposed definition of the disturbance energy of the vortex wake can be verified by experiments on an axisymmetric body at angle of attack, then what remains to be done is the development of a stability analysis of the vortex wake which has this energy as an output, so that the energy can be calculated and the effects of various parameters on the energy studied.

VII. Conclusions

A new definition of the disturbance energy contained in an asymmetric vortex wake has been presented. This energy is the product of the section side force coefficient, the vortex separation distance, and the vortex angle. The behavior of the disturbance energy so defined has been investigated using the potential flow past a circular cylinder with two vortices as a “virtual wind tunnel” for the flow. The disturbance energy so defined has exhibited the behaviors associated with the asymmetric vortex wake of an axisymmetric body at angle of attack, and so may be used in experimental investigations of the parameters affecting the stability of the vortex wake of the body at angle of attack.

References

Figures

Fig. 1 Variation of side force coefficient with unsteady bleed coefficient for cone-cylinder of Williams and Bernhardt; left figure $\alpha = 45$ deg, right figure $\alpha = 55$ deg (Fig. 4 in Ref. 5)

Fig. 2 Growth of wake distortion energy in experiments on cone-cylinder at $\alpha = 45$ deg (Fig. 17 in Ref. 7)

Fig. 3 Schematic of vortex wake cross section (Fig. 6 in Ref. 9)
Fig. 4 Vortex wake geometry definitions for potential flow analysis (Fig. 6.2 in Ref. 10)

Fig. 5 Föppl solution for stationary vortex positions behind circular cylinder in cross flow

Fig. 6 Example of solution for vortex trajectories – $r_0 = 2.0$, $\Delta\theta = 0.5$ deg

Fig. 7 Development of side force coefficient with time for $r_0 = 2.0$, $\beta_0 = 0.5$ deg
Fig. 8 Development of vortex angle with time for \( r_0 = 2.0, \beta_0 = 0.5 \text{ deg} \) (see Fig. 3)

Fig. 9 Development of disturbance energy with time for \( r_0 = 2.0, \beta_0 = 0.5 \text{ deg} \)

Fig. 10 Effect of initial disturbance on vortex trajectories: \( r_0 = 2.0 \)
Fig. 11  Effect of initial disturbance on side force coefficient: $r_0 = 2.0$

Fig. 12  Effect of initial disturbance on vortex angle: $r_0 = 2.0$

Fig. 13  Effect of initial disturbance on disturbance energy development: $r_0 = 2.0$
Fig. 14  Bump geometry for computational study by Levy, Hesselink and Degani (Fig. 20 in Ref. 15)

Fig. 15  Effects of bump height on axial development of sectional side force coefficient on ogive-cylinder at 40 deg angle of attack (Fig. 21 in Levy et al., Ref. 15)

Fig. 16  Effect of initial location on vortex trajectories for $\beta_0 = 0.5$ deg; “+” symbols indicate positions of vortices at maxima in wake disturbance energy
Fig. 17  Effect of initial position on side force coefficient development: $\beta_0 = 0.5$ deg

Fig. 18  Effect of initial position on wake disturbance energy development: $\beta_0 = 0.5$ deg
Fig. 19  Effect of initial position on initial wake disturbance energy development: $\beta_0 = 0.5$ deg. Linear regression curvefits and regression coefficient values are included for each $r_0$.

$$y = 1.699x - 0.3323 \quad R^2 = 0.9999$$

$$y = 0.9143x - 0.1216 \quad R^2 = 0.9998$$

$$y = 0.5706x - 0.0421 \quad R^2 = 0.9999$$

$$y = 0.3897x + 0.0137 \quad R^2 = 1$$

Fig. 20  Comparison of disturbance growth rates as function of initial vortex radial position. Solid curve is twice value given by Smith (Eqn. 11), points are slopes from Fig. 19.
Fig. 21  Flow visualization apparatus in experiments of Bridges and Hornung (Fig. 6 in Ref. 18)

Fig. 22  Vortex wake images (Fig. 13 in Ref. 9)