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A. D. Freed, J. Liao & D. R. Einstein

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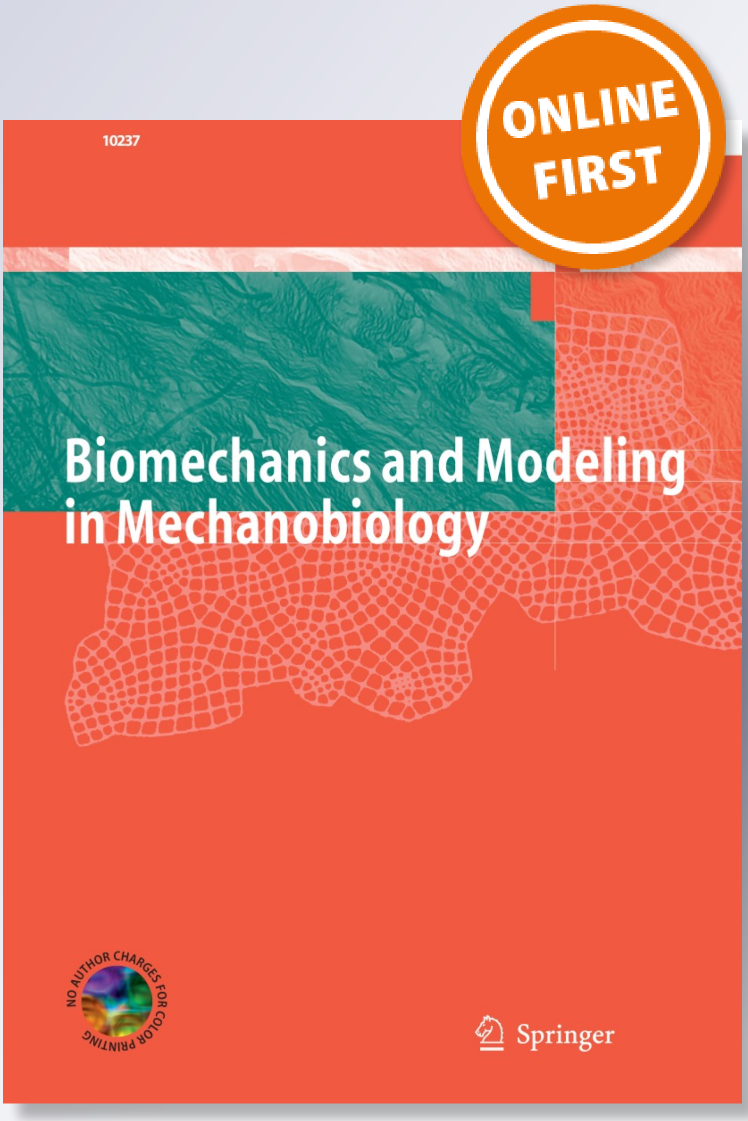

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ORIGINAL PAPER

A membrane model from implicit elasticity theory: application to visceral pleura

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Abstract A Fungean solid is derived for membranous materials as a body defined by isotropic response functions whose mathematical structure is that of a Hookean solid where the elastic constants are replaced by functions of state derived from an implicit, thermodynamic, internal energy function. The theory utilizes Biot's (Lond Edinb Dublin Philos Mag J Sci 27:468–489, 1939) definitions for stress and strain that, in one-dimension, are the stress/strain measures adopted by Fung (Am J Physiol 28:1532–1544, 1967) when he postulated what is now known as Fung's law. Our Fungean membrane model is parameterized against a biaxial data set acquired from a porcine pleural membrane subjected to three, sequential, proportional, planar extensions. These data support an isotropic/deviatoric split in the stress and strain-rate hypothesized by our theory. These data also demonstrate that the material response is highly nonlinear but, otherwise, mechanically isotropic. These data are described reasonably well by our otherwise simple, four-parameter, material model.

Keywords Biot stress · Biot strain · Deviatoric stress · Distortional strain · Finite deformation · Proportional loading · Tangent moduli

1 Introduction

A theoretical framework for implicit elastic solids was developed in a recent paper by Freed and Einstein (2013). In that document, Green strain and the second Piola-Kirchhoff stress were selected for the conjugate state variables. The theory presented in this paper is based after our earlier work, but herein Biot (1939) strain and stress have been selected as the primitive variables. After their isotropic and deviatoric constituents are established and the resulting governing constitutive equation for an implicit elastic solid is put forward, a material model is constructed and applied to the visceral pleura, the membrane encasing the lung. Prior biomechanical studies of this particular tissue have been reviewed by Humphrey et al. (1987) and Humphrey (1998).

Our theory is described in terms of the engineering stress $\mathsf{T} = \mathrm{sym}(\mathsf{R}^{\mathsf{T}}\mathsf{P})$ and strain $\mathsf{E} = \mathsf{U} - \mathsf{I}$ tensors of Biot (1939) where R and U are the rotation and (right) stretch tensors from a polar decomposition of the deformation gradient, and where P is the first Piola-Kirchhoff stress. In $\mathbb{R}^2$, these tensors are referred to as the surface tension and areal strain. Surface tension and areal strain-rate are decomposed into additive isotropic and deviatoric components, following a procedure advanced by Freed and Einstein (2013) and obey

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A. D. Freed (✉)
Department of Mechanical Engineering, Saginaw Valley State University, 202 Pioneer Hall, 7400 Bay Road, University Center, MI 48710, USA
e-mail: adfreed@svsu.edu

J. Liao
Tissue Bioengineering Laboratory, Department of Agricultural and Biological Engineering, Mississippi State University, Mississippi State, MS 39762, USA
e-mail: jliao@abe.msstate.edu

D. R. Einstein
Computational Biology & Bioinformatics, Pacific Northwest National Laboratory, 1007 Rogers Street Northwest, Olympia, WA 98502, USA
e-mail: daniel.einstein@pnnl.gov

$$\mathsf{T} = \bar{\mathsf{T}} + \tau\,\mathsf{U}^{-1} \quad \text{where} \quad \tau = \tfrac{1}{2}\,\mathrm{tr}(\mathsf{T}\mathsf{U}), \tag{1}$$

$$\mathrm{d}\mathsf{E} = \mathrm{d}\bar{\mathsf{E}} + \tfrac{1}{2}\,\mathrm{d}\eta\,\mathsf{U} \quad \text{where} \quad \mathrm{d}\eta = \mathrm{tr}\left(\mathsf{U}^{-1}\,\mathrm{d}\mathsf{E}\right) \tag{2}$$

so that $\mathrm{tr}(\bar{\mathsf{T}}\,\mathsf{U}) = 0$ and $\mathrm{tr}\left(\mathsf{U}^{-1}\,\mathrm{d}\bar{\mathsf{E}}\right) = 0$. Here U and U^{-1} serve as the covariant and contravariant metrics of deformation, just as C $(= \mathsf{U}^2)$ and C^{-1} $(= \mathsf{U}^{-2})$ served as the covariant and contravariant metrics of deformation in our prior study (Freed and Einstein 2013), and wherein Green strain and the second Piola-Kirchhoff stress were selected for establishing strain and stress. Scalars τ and $\mathrm{d}\eta$ are the isotropic variables quantifying surface pressure and the areal rate of dilatation. Tensors $\bar{\mathsf{T}}$ and $\mathrm{d}\bar{\mathsf{E}}$ are the deviatoric variables quantifying surface tension and areal strain-rate, which are traceless because of the definitions for τ and $\mathrm{d}\eta$.[1] Tensor $\bar{\mathsf{E}} = \int \mathrm{d}\bar{\mathsf{E}}$ is *not* deviatoric; it is called areal distortion.

A first-order differential equation is used to describe the constitutive response of our theory, wherein stress-rate is a homogeneous function in strain-rate and therefore independent of time (Noll 1955; Freed and Einstein 2013), viz.,

$$\mathrm{d}\mathsf{T} = \mathbb{M} : \mathrm{d}\mathsf{E} \tag{3}$$

where, as Noll has pointed out, such constitutive models can have responses that depend upon the paths traveled in state space. For materials that can be described by the isotropic/deviatoric splits of Eqs. (1, 2), the tangent modulus $\mathbb{M}$ has a general tensorial structure of

$$\begin{aligned}\mathbb{M} = {} & \frac{\mathrm{d}\tau}{\mathrm{d}\eta}\,\mathsf{U}^{-1} \otimes \mathsf{U}^{-1} - \tau\,\mathsf{U}^{-1} \odot \mathsf{U}^{-1} \\ & + \frac{\mathrm{d}\bar{\mathsf{T}}}{\mathrm{d}\bar{\mathsf{E}}} : \left(\mathsf{I} \odot \mathsf{I} - \tfrac{1}{2}\,\mathsf{U} \otimes \mathsf{U}^{-1}\right),\end{aligned} \tag{4}$$

where $\otimes$ and $\odot$ are the outer and inner tensor products. Specific formulæ for τ, $\mathrm{d}\tau/\mathrm{d}\eta$, and $\mathrm{d}\bar{\mathsf{T}}/\mathrm{d}\bar{\mathsf{E}}$ will establish specific constitutive responses for suitable material descriptions.

The constitutive structure of Eqs. (3, 4) is general. It applies for both elastic and inelastic materials whenever Biot's measures for strain and stress are adopted. For an implicit, Rajagopal, elastic solid of a rate type described by isotropic and deviatoric energy functions $\mathfrak{W}_i(\eta, \tau)$ and $\mathfrak{W}_d(\bar{\mathsf{E}}, \bar{\mathsf{T}})$, derived following a methodology put forward by Freed and Einstein (2013), one is eventually led to

$$\frac{\mathrm{d}\tau}{\mathrm{d}\eta} = \left(\frac{\partial^2 \mathfrak{W}_i(\eta, \tau)}{\partial \tau^2}\right)^{-1} \left(1 - \frac{\partial^2 \mathfrak{W}_i(\eta, \tau)}{\partial \tau\, \partial \eta}\right) \tag{5}$$

and

$$\frac{\mathrm{d}\bar{\mathsf{T}}}{\mathrm{d}\bar{\mathsf{E}}} = \left(\frac{\partial^2 \mathfrak{W}_d\left(\bar{\mathsf{E}}, \bar{\mathsf{T}}\right)}{\partial \bar{\mathsf{T}}\, \partial \bar{\mathsf{T}}}\right)^{-1} : \left(\mathsf{I} \odot \mathsf{I} - \frac{\partial^2 \mathfrak{W}_d\left(\bar{\mathsf{E}}, \bar{\mathsf{T}}\right)}{\partial \bar{\mathsf{T}}\, \partial \bar{\mathsf{E}}}\right) \tag{6}$$

where the thermodynamic potential functions $\mathfrak{W}_i(\eta, \tau)$ and $\mathfrak{W}_d(\bar{\mathsf{E}}, \bar{\mathsf{T}})$ are implicit because they depend upon both conjugate variables, viz., strain and stress. There is a proper thermodynamic basis for implicit elasticity (Rajagopal 2003) and such theories allow a linearized strain to be a nonlinear function of stress, an important feature for modeling biological materials.

The implicit theory of elasticity constructed here and that of Rajagopal and Srivinasa (2007) are described by differential equations, e.g., Eqs. (3–6). This is not requisite of implicit elasticity. The original elastic theories of Rajagopal (2003) are implicit descriptions expressed in terms of stress and strain. These two mathematical approaches are not totally equivalent. Each has unique features that can be brought to bear when constructing a model for a material.

2 Models

The Cayley-Hamilton theorem applied to a 2×2 matrix, say M, requires

$$\mathsf{M}^2 - \mathrm{tr}\,(\mathsf{M})\;\mathsf{M} + \tfrac{1}{2}\left((\mathrm{tr}\mathsf{M})^2 - \mathrm{tr}\left(\mathsf{M}^2\right)\right)\mathsf{I} = \mathsf{0}$$

where it is readily verified that $\frac{1}{2}((\mathrm{tr}\mathsf{M})^2 - \mathrm{tr}(\mathsf{M}^2)) = \det \mathsf{M}$. For two separate 2×2 matrices, the Cayley-Hamilton theorem requires

$$\begin{aligned}&\mathsf{M}\mathsf{N} + \mathsf{N}\mathsf{M} - \mathrm{tr}\,(\mathsf{M})\;\mathsf{N} - \mathrm{tr}\,(\mathsf{N})\;\mathsf{M} \\ &\quad + \left(\mathrm{tr}\,(\mathsf{M})\;\mathrm{tr}\,(\mathsf{N}) - \mathrm{tr}\,(\mathsf{M}\mathsf{N})\right)\mathsf{I} = \mathsf{0}\end{aligned}$$

for any M and N. The latter reverts to the former whenever $\mathsf{M} = \mathsf{N}$. Additional formulæ arising from the Cayley-Hamilton theorem pertaining to three separate 2×2 matrices can be found in, e.g., Spencer (1971 pp. 296–7).

From the above, derived from the Cayley-Hamilton theorem, any isotropic response function that depends upon a single, mixed, tensor field in $\mathbb{R}^2$, say M, can be described in terms of at most two, independent, scalar invariants, viz.,

$$I_1 = \mathrm{tr}(\mathsf{M}) \quad \text{and} \quad I_2 = \det(\mathsf{M}) = \tfrac{1}{2}\left((\mathrm{tr}\mathsf{M})^2 - \mathrm{tr}\left(\mathsf{M}^2\right)\right). \tag{7}$$

Any isotropic function expressed in terms of two, distinct, mixed, tensor fields in $\mathbb{R}^2$ has five independent invariants:

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[1] Deviatoric, as the terminology is used here, implies that the trace of a field is zero when contracted with its metric of deformation, in a geometric sense. Whenever one employs the thermodynamic conjugate fields of Almansi strain and Cauchy stress, Eulerian fields, it is well known that the metric of deformation is simply the identity tensor I. Whenever one employs the conjugates of Green strain and the second Piola-Kirchhoff stress, Lagrangian fields, it is well known that the metric of deformation is the right Cauchy-Green deformation tensor C. Likewise, whenever one employs Biot's conjugate fields for stress and strain, the metric of deformation for tensor contraction is the stretch tensor U. This fact is less known, because Biot's measures for stress and strain are less widely used.

those of Eq. (7) pertaining to each tensor, say M and N, and one additional invariant arising from their interaction, viz.,

$$I_1 = \mathrm{tr}(\mathsf{M}), \quad I_2 = \det(\mathsf{M}), \quad I_3 = \mathrm{tr}(\mathsf{N}), \quad I_4 = \det(\mathsf{N}),$$
$$I_5 = \mathrm{tr}(\mathsf{M})\,\mathrm{tr}(\mathsf{N}) - \mathrm{tr}(\mathsf{MN}). \tag{8}$$

Two-dimensional, implicit, elastic models are described in terms of these five invariants. Three-dimensional, implicit, elastic models require ten invariants (Spencer 1971; Freed and Einstein 2013). A significant reduction in complexity is afforded whenever one adopts the Wong and Shield (1969) approach of using invariants defined over $\mathbb{R}^2$ to construct a membrane model. One expects the additional invariants over $\mathbb{R}^3$, when specialized for membranes, will produce higher-order effects in the response, effects that are neglected in this initial study.

2.1 Hookean membrane

The implicit, isotropic, Hookean membrane has a functional dependence described in terms of the surface pressure τ and its two invariants for surface tension $\bar{\mathsf{T}}$ that, in accordance with Eq. (7), take on forms of[2]

$$I_1 = \mathrm{tr}\left(\bar{\mathsf{T}}\mathsf{U}\right) = 0 \quad \text{and} \quad I_2 = -\det\left(\bar{\mathsf{T}}\mathsf{U}\right) = \tfrac{1}{2}\,\mathrm{tr}\left(\bar{\mathsf{T}}\mathsf{U}\bar{\mathsf{T}}\mathsf{U}\right) \tag{9}$$

where U enters into the construction as the metric of deformation used for indical contraction, e.g., $\mathrm{tr}(\bar{\mathsf{T}}\mathsf{U}\bar{\mathsf{T}}\mathsf{U}) = \bar{T}^{IJ} U_{JK} \bar{T}^{KL} U_{LI}$. Recall that the first invariant is zero because the deviatoric surface tension is traceless, by definition, cf. Eq. (1). The classic formulation of linear elasticity is captured in terms of the remaining second invariant by considering the quadratic energy functions

$$\mathfrak{W}_i(\tau) = \tau^2/2E \tag{10}$$
$$\mathfrak{W}_d\left(\bar{\mathsf{T}}\right) = I_2/2\mu = \mathrm{tr}\left(\bar{\mathsf{T}}\mathsf{U}\bar{\mathsf{T}}\mathsf{U}\right)/4\mu \tag{11}$$

where E and μ are the elastic and shear moduli. The elastic modulus in $\mathbb{R}^2$ corresponds to $E = A\,\mathrm{d}\tau/\mathrm{d}A$, A being an element of area experiencing a uniform surface pressure τ over a region in $\mathbb{R}^2$. This contrasts with the bulk modulus κ from classical elasticity where $\kappa = V\,\mathrm{d}P/\mathrm{d}V$, V being an element of volume experiencing a uniform bulk pressure P over a region in $\mathbb{R}^3$.

Inserting the isotropic energy function (10) into its governing constitutive Eq. (5) leads to

$$\mathrm{d}\tau = \left(\frac{1}{E}\right)^{-1} \mathrm{d}\eta = E\,\mathrm{d}\eta \quad \therefore \quad \tau = E\,\eta \tag{12}$$

while substituting its associated deviatoric energy function (11) into its governing constitutive Eq. (6) leads to

$$\mathrm{d}\bar{\mathsf{T}} = \left(\frac{1}{2\mu}\,\mathsf{U} \odot \mathsf{U}\right)^{-1} : \mathrm{d}\bar{\mathsf{E}} = 2\mu\,\mathsf{U}^{-1} \odot \mathsf{U}^{-1} : \mathrm{d}\bar{\mathsf{E}}. \tag{13}$$

When these two constitutive formulæ are placed into Eq. (4), the tangent modulus $\mathbb{M}$ becomes

$$\mathbb{M} = (2\mu - \tau)\,\mathsf{U}^{-1} \odot \mathsf{U}^{-1} + (E - \mu)\,\mathsf{U}^{-1} \otimes \mathsf{U}^{-1}. \tag{14}$$

This modulus reduces to the tangent modulus of a classic Hookean solid under conditions of infinitesimal deformation: when it is in a state of plane stress, when stretches are infinitesimal, i.e., $\mathsf{U} \approx \mathsf{I}$, and when there is minimal pressure, viz., $|\tau| \ll \mu$. Under these conditions $\mathbb{M}$ simplifies to $2\mu\,\mathsf{I} \odot \mathsf{I} + (E - \mu)\,\mathsf{I} \otimes \mathsf{I}$, which is the classic Hookean solid from linear elasticity. In plane stress, $E - \mu$ is the Lamé constant that, in three-dimensional analysis, becomes $\lambda = \kappa - \frac{2}{3}\mu$ with κ being the bulk modulus.

The inverse to the symmetric tangent modulus $\mathbb{M}$ is a symmetric tangent compliance, i.e., $\mathbb{C} = \mathbb{M}^{-1}$, that for an isotropic Hookean membrane becomes[3]

$$\mathbb{C} = \frac{1}{2\mu - \tau}\left(\mathsf{U} \odot \mathsf{U} - \frac{E - \mu}{2E - \tau}\,\mathsf{U} \otimes \mathsf{U}\right). \tag{15}$$

In the classic Hookean solid, under a state of plane stress, the leading coefficient in $\mathbb{C}$ would be $(1 + \nu)/E$ where ν is Poisson's ratio, which is a material property in a linear elastic solid. (It is a response function in nonlinear materials.) Equating coefficients leads one to the observation

$$\nu = \frac{E - 2\mu + \tau}{2\mu - \tau}$$

that becomes $\nu = (E - 2\mu)/2\mu$ whenever $\tau = 0$, agreeing with its interpretation in linear elasticity. Poisson's ratio is therefore a response function in our finite theory of elasticity; it is modulated by pressure τ.

The response of a Hookean membrane will be stable in the sense of Hill (1957) provided that $\mathrm{d}\mathsf{E} : \mathbb{M} : \mathrm{d}\mathsf{E} > 0$ whenever $\mathrm{d}\mathsf{E} \neq \mathsf{0}$, which leads to

$$(E - \mu)\left(\mathrm{tr}\left(\mathsf{U}^{-1}\mathrm{d}\mathsf{E}\right)\right)^2 + 2\mu\,\mathrm{tr}\left(\mathsf{U}^{-1}\mathrm{d}\mathsf{E} \cdot \mathsf{U}^{-1}\mathrm{d}\mathsf{E}\right)$$
$$> \tau\,\mathrm{tr}\left(\mathsf{U}^{-1}\mathrm{d}\mathsf{E} \cdot \mathsf{U}^{-1}\mathrm{d}\mathsf{E}\right) \tag{16}$$

given that $\mathrm{d}\mathsf{E} \neq \mathsf{0}$. Instability becomes a possibility whenever the surface pressure τ exceeds, in a certain sense, the elastic cohesion of the material, approximately $E + \mu$, at which point the material would disintegrate. This outcome of our nonlinear Hookean solid does not follow from linear elasticity.

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[2] The sign of the active invariant has been switched to yield positive values during extensions.

[3] To verify $\mathbb{C} : \mathbb{M} = \mathsf{I} \odot \mathsf{I}$ requires identities: $\mathsf{U}^{-1} \otimes \mathsf{U}^{-1} : \mathsf{U} \otimes \mathsf{U} = 2\,\mathsf{U}^{-1} \otimes \mathsf{U}$, $\mathsf{U}^{-1} \otimes \mathsf{U}^{-1} : \mathsf{U} \odot \mathsf{U} = \mathsf{U}^{-1} \otimes \mathsf{U}$, $\mathsf{U}^{-1} \odot \mathsf{U}^{-1} : \mathsf{U} \otimes \mathsf{U} = \mathsf{U}^{-1} \otimes \mathsf{U}$ and $\mathsf{U}^{-1} \odot \mathsf{U}^{-1} : \mathsf{U} \odot \mathsf{U} = \mathsf{I} \odot \mathsf{I}$.

2.2 Fungean membrane

A general, implicit, elastic solid admits a potential function for internal energy that depends upon both stress and strain (Rajagopal 2003). As such, in accordance with Eqs. (1, 2, 8), an isotropic, implicit, elastic membrane defined over $\mathbb{R}^2$ can depend upon, at most, the surface pressure τ, the dilatation η, and the following five invariants[4]

$$\begin{aligned} &I_1 = \operatorname{tr}\left(\bar{\mathsf{T}}\mathsf{U}\right) = 0, \quad I_2 = \tfrac{1}{2}\operatorname{tr}\left(\bar{\mathsf{T}}\mathsf{U}\bar{\mathsf{T}}\mathsf{U}\right), \\ &I_3 = -\operatorname{tr}\left(\mathsf{U}^{-1}\bar{\mathsf{E}}\right), \quad I_4 = -\det\left(\mathsf{U}^{-1}\bar{\mathsf{E}}\right), \\ &I_5 = \operatorname{tr}\left(\bar{\mathsf{T}}\bar{\mathsf{E}}\right) \end{aligned} \tag{17}$$

where knowledge of $\bar{\mathsf{T}}$ being deviatoric, and hence of $I_1 = 0$, and of $\bar{\mathsf{E}}$ not being deviatoric, and hence of $I_3 \neq 0$, has been used to reduce this set of invariants to its simplest form.

A Fung elastic solid, as defined by (Freed and Einstein, 2013, Defn. 4), is any implicit elastic solid whose mixed derivatives in stress and strain, namely $\partial^2\mathfrak{W}_i/\partial\tau\,\partial\eta$ and $\partial^2\mathfrak{W}_d/\partial\bar{\mathsf{T}}\,\partial\bar{\mathsf{E}}$ in Eqs. (5, 6), are first-order in their respective measures for stress. A Fung elastic solid, so defined, has a natural coupling between its compliance contribution, i.e., $\partial^2\mathfrak{W}_i/\partial\tau^2$ and $\partial^2\mathfrak{W}_d/\partial\bar{\mathsf{T}}\,\partial\bar{\mathsf{T}}$, and its Fungean effect, viz., $\partial^2\mathfrak{W}_i/\partial\tau\,\partial\eta$ and $\partial^2\mathfrak{W}_d/\partial\bar{\mathsf{T}}\,\partial\bar{\mathsf{E}}$, that arises from the potential structure of our theory. Freed and Einstein (2013) call this coupling the Rajagopal effect in honor of Rajagopal's (2003) seminal work in implicit elasticity.

The simplest Fungean membrane that one can conceive adds a coupled term to each of its two, quadratic, energy functions (10, 11) describing a Hookean membrane. These additional terms are first-order in areal strain and second-order in surface tension, in accordance with our definition of a Fungean solid. The only admissible combination of invariants that permits this extra term in the deviatoric energy is $I_2 I_3$. Consequently, one is lead, in a natural way, to consider potential functions for the stored elastic energy of

$$\mathfrak{W}_i(\eta, \tau) = \frac{\tau^2}{2E}\,(1 - \alpha\eta)\,, \tag{18}$$

$$\begin{aligned} \mathfrak{W}_d(\bar{\mathsf{E}}, \bar{\mathsf{T}}) &= \frac{I_2}{2\mu}\,(1 - \beta I_3) \\ &= \frac{1}{4\mu}\operatorname{tr}\left(\bar{\mathsf{T}}\mathsf{U}\bar{\mathsf{T}}\mathsf{U}\right)\left(1 + \beta\operatorname{tr}\left(\mathsf{U}^{-1}\bar{\mathsf{E}}\right)\right) \end{aligned} \tag{19}$$

where α and β are coined the Fung isotropic and deviatoric parameters in honor of Fung's (1967) 1D empirical law: $\mathrm{d}T = E(1 + \hat{\alpha}T)\,\mathrm{d}\lambda$ where T is traction and λ is stretch.[5] Parameters α and β are dimensionless. This model has two parameters to describe each energy contribution; hence, it is a four-parameter model.

These two energy functions introduce the notion of a limiting state of areal strain. This idea originates, to the best of our knowledge, with a 1D material model introduced by Carton et al. (1962). Their model was expressed in terms of a compliance. Later, Fung (1967) introduced an equivalent 1D model expressed in terms of a modulus. Today, this notion is widely referred to as Fung's law.

When the isotropic energy function in Eq. (18) is substituted into its governing constitutive Eq. (5), an evolution equation for surface pressure or, alternatively, for areal dilatation ensues, viz.,

$$\mathrm{d}\tau = \frac{E + \alpha\tau}{1 - \alpha\eta}\,\mathrm{d}\eta \quad \text{or, equivalently,} \quad \mathrm{d}\eta = \frac{1 - \alpha\eta}{E + \alpha\tau}\,\mathrm{d}\tau \tag{20}$$

depending upon which variable is the independent variable of the boundary value problem being considered. This is a separable, first-order, differential equation whose solution is $(E + \alpha\tau)(1 - \alpha\eta) = E$ so that

$$\tau = \frac{E\,\eta}{1 - \alpha\eta} \quad \text{or} \quad \eta = \frac{\tau}{E + \alpha\tau} \tag{21}$$

depending upon which variable is being controlled. Dilatation is strain-limited in the sense of Carton et al. (1962), because $\tau \to \infty$ as $\eta \to 1/\alpha$ from below.

Substituting the deviatoric energy function (19) into its governing constitutive Eq. (6) produces an evolution equation for the deviatoric surface tension of

$$\mathrm{d}\bar{\mathsf{T}} = \frac{2\mu}{1 + \beta\operatorname{tr}\left(\mathsf{U}^{-1}\bar{\mathsf{E}}\right)}\,\mathsf{U}^{-1}\,\mathrm{d}\bar{\mathsf{E}}\,\mathsf{U}^{-1} \tag{22}$$

where the fact that $\bar{\mathsf{T}} \otimes \mathsf{U}^{-1} : \mathrm{d}\bar{\mathsf{E}} = \operatorname{tr}(\mathsf{U}^{-1}\,\mathrm{d}\bar{\mathsf{E}})\,\bar{\mathsf{T}} = \mathsf{0}$ has been used, which vanishes because of Eq. (2). Equation (22) is traceless, $\operatorname{tr}(\mathsf{U}\,\mathrm{d}\bar{\mathsf{T}}) = 0$, because $\operatorname{tr}(\mathsf{U}^{-1}\mathrm{d}\bar{\mathsf{E}}) = 0$ by definition. This is consistent with the very fact that $\operatorname{tr}(\mathsf{U}\,\bar{\mathsf{T}}) = 0$ via its definition in Eq. (1).

Collectively, these two constitutive functions combine with Eq. (3) to produce a symmetric tangent modulus for this Fungean membrane of

$$\mathbb{M} = (2\tilde{\mu} - \tau)\,\mathsf{U}^{-1} \odot \mathsf{U}^{-1} + \left(\tilde{E} - \tilde{\mu}\right)\mathsf{U}^{-1} \otimes \mathsf{U}^{-1} \tag{23}$$

whose inverse is the symmetric tangent compliance

$$\mathbb{C} = \frac{1}{2\tilde{\mu} - \tau}\left(\mathsf{U} \odot \mathsf{U} - \frac{\tilde{E} - \tilde{\mu}}{2\tilde{E} - \tau}\,\mathsf{U} \otimes \mathsf{U}\right). \tag{24}$$

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[4] The sign of these invariants have been manipulate to yield positive values during extensions.

[5] This paper presents the first nonlinear theory of elasticity, known to the authors, that incorporates the engineering stress and strain measures of Biot (1939) into a material model of the Fungean type that, in one-dimension, constitutes the actual measures for stress and strain used by Fung (1967) in his empirical model/law.

wherein

$$\tilde{E} = \frac{E + \alpha\tau}{1 - \alpha\eta} = \frac{E}{(1 - \alpha\eta)^2} = \frac{(E + \alpha\tau)^2}{E} \tag{25}$$

and

$$\tilde{\mu} = \frac{\mu}{1 - \beta\, I_3} = \frac{\mu}{1 + \beta\, \mathrm{tr}\left(\mathsf{U}^{-1}\bar{\mathsf{E}}\right)} \tag{26}$$

are *elastic tangent moduli*, with τ and η following from Eq. (21), and $\bar{\mathsf{E}}$ from integrating $\int_0^t (\mathrm{d}\bar{\mathsf{E}}(t')/\mathrm{d}t')\,\mathrm{d}t'$ with $\mathrm{d}\bar{\mathsf{E}}$ being established in Eq. (2).

Equations (3, 23, 25, 26) comprise an implicit elastic membrane model requiring four material parameters: E, μ, α, and β. A deformation of equibiaxial extension will be independent of parameters μ and β, while a deformation of simple shear will be independent of parameters E and α. This Fungean membrane reduces to our Hookean membrane whenever $\alpha \approxeq \beta \approxeq 0$ or, equivalently, whenever $\tilde{E} \approxeq E$ and $\tilde{\mu} \approxeq \mu$.

In terms of their modulus and compliance tensors, the single feature that distinguishes this four-parameter Fungean membrane from the two-parameter Hookean membrane that it generalizes is: *elastic constants E and μ in the Hookean model have been replaced with elastic tangent functions $\tilde{E}$ and $\tilde{\mu}$ in the Fungean model*. These functions were derived from theory, firmly based upon thermodynamic principles. *They are not ad hoc!* The authors expect this feature will strongly affect the future utility of our theory when modeling biological tissues, and even some man-made elastomers (cf. Freed 2013). A similar result was acquired by Rajagopal (2003) where he derived a Navier-Stokes model with a pressure dependent viscosity from a thermodynamically admissible, implicit constitutive construction.

The material response of this Fungean membrane will be stable whenever Eq. (16) is satisfied, with the elastic tangent moduli $\tilde{E}$ and $\tilde{\mu}$ replacing the elastic material constants E and μ found therein.

3 Proportional biaxial experiments

In a proportionally loaded planar membrane that does not experience rotation, tractions are imposed on specimens at a fixed ratio of ϕ causing the components of the Lagrangian surface tension (Biot (1939) stress in 2D) to be described by

$$\mathsf{T} = \begin{bmatrix} T & 0 \\ 0 & \phi T \end{bmatrix}, \qquad T \geq 0 \quad \text{and} \quad 0 \leq \phi \leq 1, \tag{27}$$

with axial tensions of $T_1(t) = T = f_1(t)/\ell_2(0)$ and $T_2(t) = \phi T = f_2(t)/\ell_1(0)$ and axial stretches of $\lambda_1 = \ell_1(t)/\ell_1(0)$ and $\lambda_2 = \ell_2(t)/\ell_2(0)$, wherein $f_1(t)$ and $f_2(t)$ are the current forces being applied in the 1- and 2-directions acting normal to initial lengths of line of $\ell_2(0)$ and $\ell_1(0)$ that parallel the 2- and 1-directions. The constant of proportionality, ϕ, is truly a constant only in terms of the components of surface tension T. Unless the material response is linear, no like constant of proportionality will exist between components of other stress measures (except the first Piola-Kirchhoff stress) nor of the various deformation fields.

In this regard, areal strain (Biot's (1939) strain measure $\mathsf{E} = \mathsf{U} - \mathsf{I}$ in 2D) and its rate have components of

$$\mathsf{E} = \begin{bmatrix} \lambda_1 - 1 & 0 \\ 0 & \lambda_2 - 1 \end{bmatrix} \quad \text{and} \quad \mathrm{d}\mathsf{E} = \begin{bmatrix} \mathrm{d}\lambda_1 & 0 \\ 0 & \mathrm{d}\lambda_2 \end{bmatrix} \tag{28}$$

the latter decomposing, according to Eq. (2), into an areal rate of dilatation of

$$\mathrm{d}\eta = \frac{\mathrm{d}\lambda_1}{\lambda_1} + \frac{\mathrm{d}\lambda_2}{\lambda_2} \quad \text{integrating to} \quad \eta = \ln \lambda_1 + \ln \lambda_2 \tag{29}$$

and a deviatoric strain-rate of

$$\mathrm{d}\bar{\mathsf{E}} = \frac{1}{2}\begin{bmatrix} \mathrm{d}\lambda_1 - \lambda_1\, \mathrm{d}\lambda_2/\lambda_2 & 0 \\ 0 & \mathrm{d}\lambda_2 - \lambda_2\, \mathrm{d}\lambda_1/\lambda_1 \end{bmatrix} \tag{30}$$

that integrates to

$$\bar{\mathsf{E}} = \frac{1}{2}\begin{bmatrix} \lambda_1 - 1 & 0 \\ 0 & \lambda_2 - 1 \end{bmatrix} - \frac{1}{2}\int_0^t \begin{bmatrix} \lambda_1(t')\,\mathrm{d}\lambda_2(t')/\lambda_2(t') & 0 \\ 0 & \lambda_2(t')\,\mathrm{d}\lambda_1(t')/\lambda_1(t') \end{bmatrix} \tag{31}$$

where it readily follows that $\mathrm{tr}(\mathsf{U}^{-1}\mathrm{d}\bar{\mathsf{E}}) = 0$, even though $\mathrm{tr}(\mathsf{U}^{-1}\bar{\mathsf{E}}) \neq 0$. The integral appearing in the above expression for $\bar{\mathsf{E}}$ is coupled between its orthogonal stretches λ_1 and λ_2 and therefore cannot be solved, analytically, except for the important special case where $\lambda_1 = \lambda_2$, i.e., equibiaxial extension. A Poisson response arises from our theory for strain. It is independent of constitutive expression.

4 Experiments on porcine visceral pleura

Fresh porcine lungs obtained from juvenile pigs from a local slaughterhouse were transported to the laboratory in phosphate buffered saline (PBS) on ice. The visceral pleura (the membrane encasing the lung) from the posterior right middle lobe of the lung was dissected and trimmed into a 25 mm × 25 mm square sample with one edge aligned along the longitudinal and the other along the circumferential. Two loops of 000 polyester suture were then attached to each side of the square sample via four stainless steel hooks (a total of eight suture loops used). The square pleural membrane sample was mounted into a custom mechanical testing system. Biaxial stretching was implemented with the samples immersed in PBS (pH 7.4) at room temperature. A detailed description of the biaxial test system can be found in Grashow et al. (2006) and Liao et al. (2008).

Briefly, the orthogonal forces were monitored by two load cells. Four markers were placed in the center of the membrane sample for real-time tissue deformation tracking using a CCD camera (cf., e.g., Sacks 2000; Humphrey 2002). Lagrangian membrane tension (force per unit initial length) was used for load control. No spatial rotations of the specimens were observed. A trial experiment found that the pleural membrane was able to withstand tension levels up to 35 N/m before tearing took place at the hook sites; thus, a target maximum tension was set at 30 N/m.

An equibiaxial preconditioning protocol took place over 10 cycles ranging between 0:0 and 30:30 N/m, after which the tension-stretch curves were observed to be repeatable. The following three protocols were then executed sequentially. For the equibiaxial protocol, the pleural membrane was loaded to a tension level of $T_{\text{longitudinal}}{:}T_{\text{circumferential}} = 30{:}30$ N/m. For proportional biaxial protocols (non-equibiaxial), the pleura was loaded to tension levels $T_{\text{longitudinal}}{:}T_{\text{circumferential}} = 30{:}15$ N/m and 15:30 N/m. To illustrate the difficulty in performing such experiments, the mean constants of proportionality controlled for the animal 2 protocols were $\phi(30{:}30) = 0.955 \pm 0.122$, $\phi(30{:}15) = 0.557 \pm 0.323$, and $\phi(15{:}30) = 0.507 \pm 0.104$, with reported errors of ± 1 standard deviation (SD), and where $0 \le \phi \le 1$. The greatest variability existed at the lowest states of tension, viz., those well within the compliant toe region.

4.1 Data

The Hookean and Fungean models derived above were each acquired from a thermodynamic potential function that represents the internal energy stored in the material caused by external work being done on the material. In our presentation, the total work done has been decomposed into separable isotropic and deviatoric contributions. Figure 1 presents the two conjugate variables describing isotropic behavior plotted against the isotropic work done in our experiments, while Fig. 2 presents the four deviatoric invariants plotted against the deviatoric work done. For our experiments, the isotropic work that was done was about an order in magnitude greater than the deviatoric work that was done.

The data presented in Figs. 1 and 2 were quantified using formulæ taken from Eqs. (27–31). The work being done was ascertained by integrating $\mathrm{d}W_i = \tau\,\mathrm{d}\eta$ and $\mathrm{d}W_d = \mathrm{tr}(\bar{\mathsf{T}}\,\mathrm{d}\bar{\mathsf{E}})$. Distortion $\bar{\mathsf{E}}$ was integrated from $\mathrm{d}\bar{\mathsf{E}}$ according to Eqs. (30, 31). In both cases, numeric integration was accomplished using the trapezoidal rule. When integrating work over an interval, the tension terms were computed as averages over the interval, whereas the areal strain-rate terms took on their discrete values at the endpoints of the interval.

All three protocols contributed to the isotropic response in Fig. 1. The response of each variable suggests the existence of a master response curve, with reasonable variability appearing in the data. These data provide experimental evidence that the isotropic and deviatoric energies are truly separable, as our theoretical formulation supposes. If there were to be an isotropic/deviatoric coupling (but none is observed), then one would expect there to be three, distinct, experimental, response curves present in Fig. 1, one for each protocol, but that is not what we found. Within experimental noise, only one experimental response curve is observed. *Isotropic and deviatoric work are separable.* Isotropic work is observed to be sensitive to both of the physical variables, viz., τ and η, for this material.

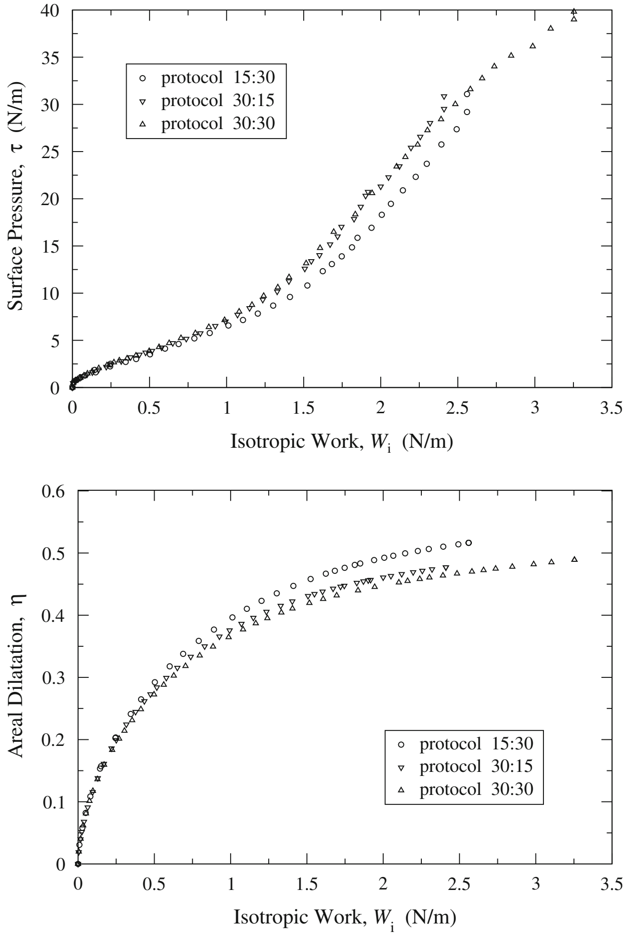

Fig. 1 Plots of the two isotropic state variables, surface pressure τ (*top*) and areal dilatation η (*bottom*), against the isotropic work done $W_i = \int \tau\,\mathrm{d}\eta$ for the 15:30, 30:15, and 30:30 protocols imposed on a porcine pleura tissue sample taken from animal 2

There is no deviatoric response in the equibiaxial experiment (the 30:30 N/m protocol) observed in Fig. 2, nor should there be according to our theory. Means for the various deviatoric response data, along with their ± 1 SD errors, are reported in the caption of Fig. 2. The two strain-invariant plots in Fig. 2, its upper-right and lower-left plots, are practically the same, and therefore, these data would be incapable of distinguishing any difference that might arise between

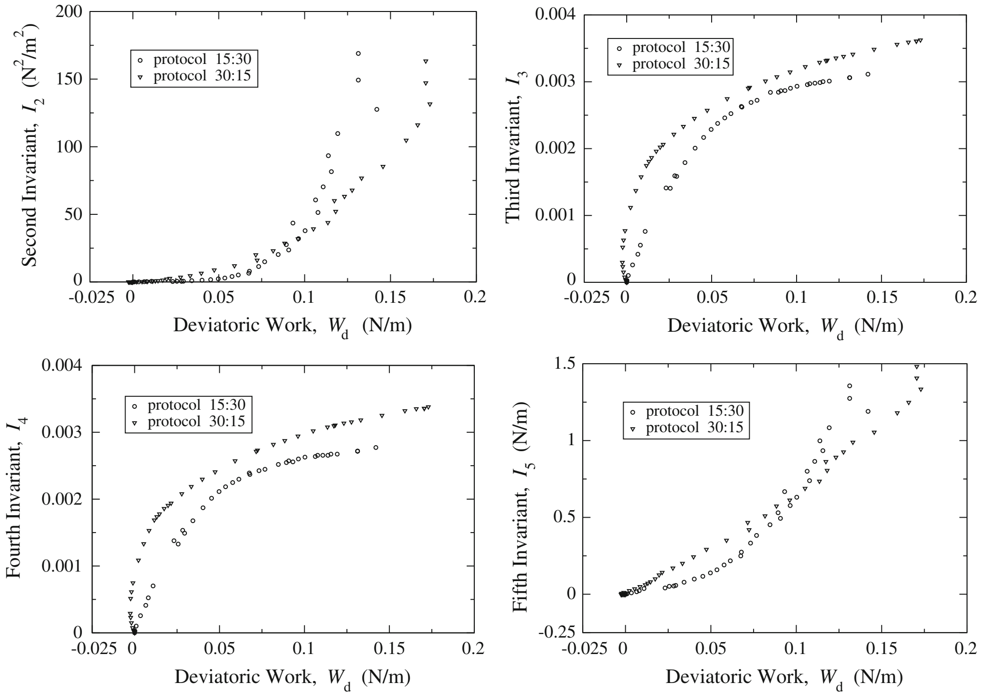

Fig. 2 Plots of the four nonzero invariants, viz., I_2 (*upper left*), I_3 (*upper right*), I_4 (*lower left*), and I_5 (*lower right*) from Eq. (17), against the deviatoric work done $W_d = \int \bar{\mathsf{T}}{:}\mathrm{d}\bar{\mathsf{E}}$ for the 15:30 and 30:15 protocols imposed on a porcine pleura tissue sample taken from animal 2. When applied to the 30:30 protocol, all data for these four deviatoric-based invariants reside around the origin of their respective plots (in accordance with theory) with means and SD errors of $W_d = -0.00117 \pm 0.00190$ N/m, $I_2 = 0.141 \pm 0.106\,\mathrm{N}^2/\mathrm{m}^2$, $I_3 = 1.32 \times 10^{-5} \pm 1.22 \times 10^{-5}$, $I_4 = 1.07 \times 10^{-5} \pm 1.26 \times 10^{-5}$, and $I_5 = 0.00175 \pm 0.00158$ N/m for these experimental data sets

invariants I_3 and I_4 in a model, where $I_3 = -\operatorname{tr}(\mathsf{U}^{-1}\bar{\mathsf{E}})$ and $I_4 = -\det(\mathsf{U}^{-1}\bar{\mathsf{E}})$ from Eq. (17). This begs the question: are I_3 and I_4 equivalent, in a theoretical sense, or is this just happenstance for these data? This remains an open question.

The 15:30 N/m protocol was the last leg run in a sequential series of three proportional extensions that comprise the loading sequence for this experimental data set. Several slip-stick events were recorded in the tangential stretch response of this last leg, causing the stretch/tension curve to retrograde several times at the lower loads. These discrete events affected the integrated results for distortion and deviatoric work; consequently, data from this leg were not used for parameterization purposes.

4.2 Fungean model characterization

The internal energy functions for a Fungean membrane, viz., Eqs. (18, 19), have parameters that quantify Hookean behavior in a neighborhood around zero tension, and asymptotic behavior, like Carton's (1962), that limits areal strain as the tension becomes unbounded. These two features of the model allow for a simple and straightforward graphical technique for parameterizing, whose outcome is recorded at the top of Table 1.

Table 1 Parameterization of the Fungean membrane model defined by Eqs. (18, 19) against experimental data for a porcine pleural membrane excised from animal 2

Work done	Moduli (N/m)	Fung parameters	Adjusted R^2
Isotropic	$E = 15$	$\alpha = 1.8$	–
Deviatoric	$\mu = 40$	$\beta = 250$	–
Isotropic	$E = 5.5$	$\alpha = 1.6$	0.982
Deviatoric	$\mu = 35$	$\beta = 250$	0.978

The top set of parameters were secured from a straightforward graphical technique. The bottom set of parameters, acquired after the first set, were obtained from a nonlinear optimization algorithm that uses an interior trust region (Coleman and Li 1996)

In a neighborhood around zero tension, the response is Hookean in that $\mathfrak{W}_i \approxeq \tau^2/2E$ and $\mathfrak{W}_d \approxeq I_2/2\mu$. Experimental data plots of τ^2 versus $W_i = \int \tau\,\mathrm{d}\eta$ and of $I_2 = \frac{1}{2}\operatorname{tr}(\bar{\mathsf{T}}\mathsf{U}\bar{\mathsf{T}}\mathsf{U})$ versus $W_d = \int \bar{\mathsf{T}} : \mathrm{d}\bar{\mathsf{E}}$ are presented in Fig. 3. The tangents in Fig. 3 were drawn by eye. Their corresponding values for E and μ are recorded at the top of Table 1.

At the most extreme states of tension, the response predicted by our Fungean model becomes asymptotic to a limiting state in areal strain, in accordance with the ideas of Carton et al. (1962). Specifically, from Eq. (18) $\eta \asymp 1/\alpha$, and from Eq. (19) $I_3 \asymp 1/\beta$. Experimental data plots of η

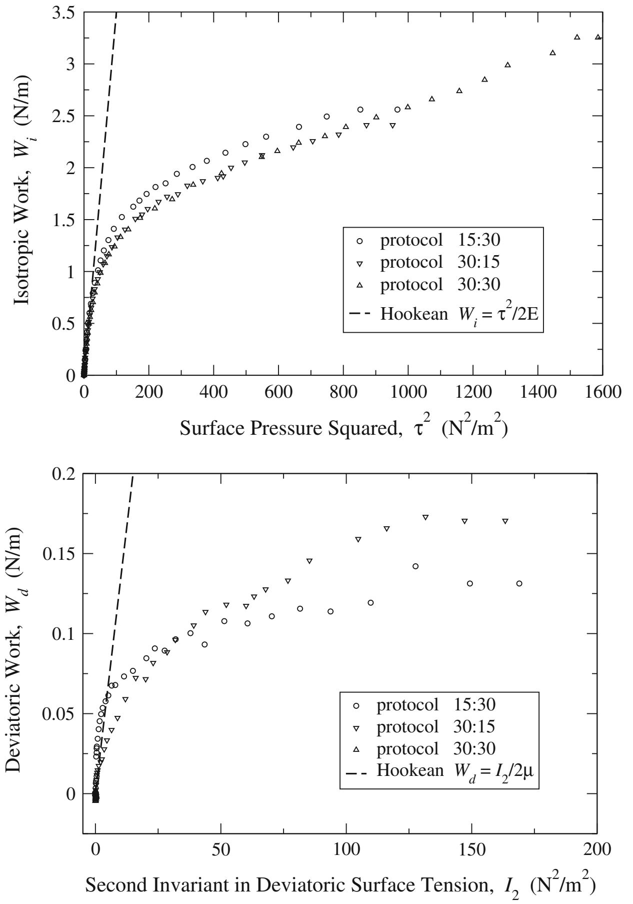

Fig. 3 Isotropic (*top*) and deviatoric (*bottom*) response plots where the work done is contrasted against an appropriate measure of tension squared. Hookean solids have data that describe *straight lines* in such figures. The initial response is Hookean. Data are for porcine pleura taken from animal 2. Parameters that associate with the slopes drawn here are recorded in the upper part of Table 1

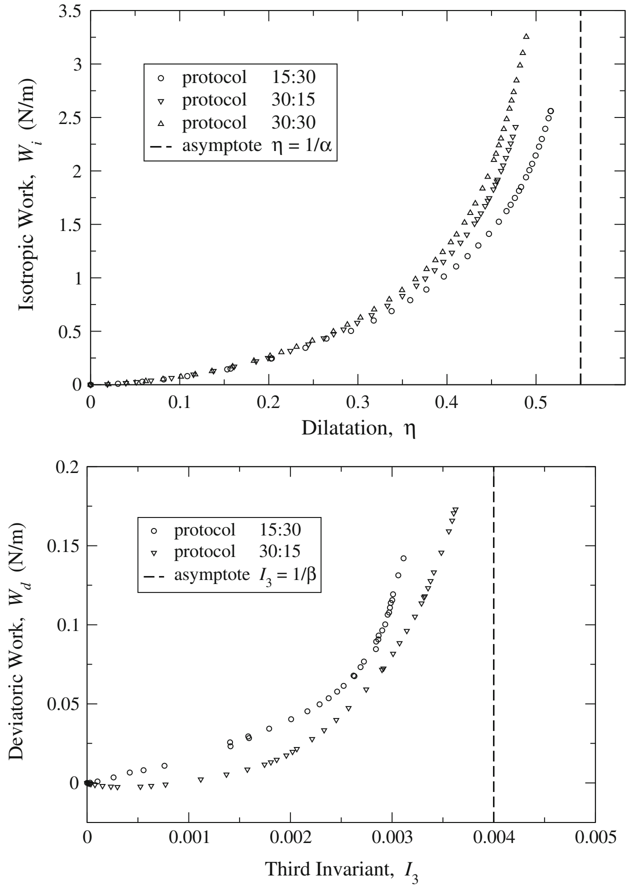

Fig. 4 Isotropic (*top*) and deviatoric (*bottom*) response plots where the work done is contrasted against an appropriate measure of areal strain. Implicit solids have data that approach an asymptotic limit in strain in the sense of Carton et al. (1962). Parameters that associate with the asymptotes drawn here are recorded in the upper part of Table 1

versus W_i and of $I_3 = -\operatorname{tr}(\mathsf{U}^{-1}\bar{\mathsf{E}})$ versus W_d are presented in Fig. 4. The asymptotes in Fig. 4 were drawn by eye. Their corresponding values for α and β are recorded at the top of Table 1.

Alternatively, after parameter estimates were made by this graphical technique, estimates were also secured using a nonlinear optimization algorithm for model parameterization. This technique employed an interior trust region, and it also allowed for parameter constraints (Coleman and Li 1996). Results from this approach to parameterization are displayed in Fig. 5, whose values are recorded in the bottom part of Table 1. They are different, yet close to those obtained graphically. Graphical values should provide reasonable estimates from which an algorithmic optimizer can be started. Our Fungean membrane model correlates these data well, especially considering there are only two parameters per response domain.

Numerical optimization techniques employed here are not ideal, because the model has no quality in it to represent how the response is supposed to transition between its two end behaviors: Hookean in the infinitesimal regime and, at the other extreme, limitless tension as areal strain approaches its asymptotic limit. Consequently, any optimizer will incorrectly adjust these model parameters, which accurately describe the two end responses, in an effort to best fit the data that lie in between. This issue, and how to best resolve it, continues to be investigated by the authors.

One could add a feature to our model that would address this transition behavior. We have resisted doing so for two reasons, although tempted. First, one would loose our simple correspondence to a Hookean membrane model where the elastic constants therein are replaced with elastic tangent functions. Second, there is so much natural variability in tissue response, sample to sample within an individual, individual to individual within a species, and species to species that any added value brought about by such model refinements is, in some sense, a vain pursuit.

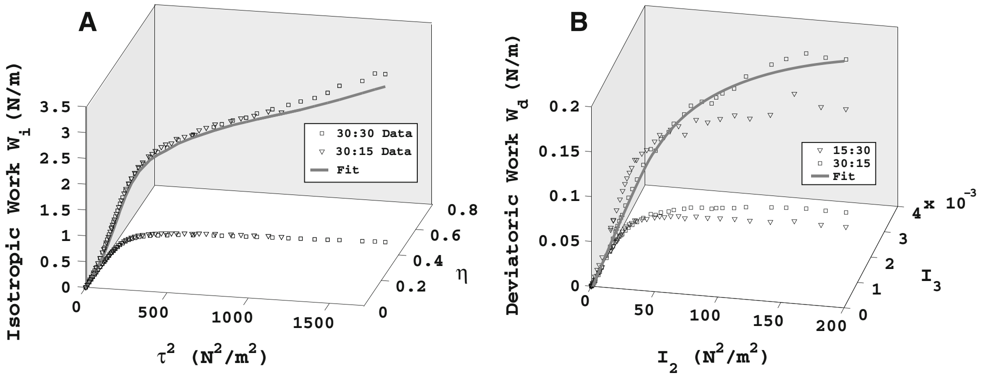

Fig. 5 The upper set of curves in each figure are the experimental contributions for the state-space responses pertaining to the isotropic (**a**) and deviatoric (**b**) energies measured from the 15:30, 30:15, and 30:30 protocols. The lower set of curves in each figure are their projections onto the τ^2 versus η and I_2 versus I_3 planes, showing a limiting state in strain, as predicted by our theory. The deviatoric response for the 30:30 protocol resides around the origin (in accordance with theory) with means and SD errors of $W_d = -0.00117 \pm 0.00190\,\text{N/m}$, $I_2 = 0.141 \pm 0.106\,\text{N}^2/\text{m}^2$, and $I_3 = 1.32 \times 10^{-5} \pm 1.22 \times 10^{-5}$ for these experimental data sets. The model fits use the parameters in the bottom part of Table 1

5 Discussion

The fact that the longitudinal and transverse responses from the equibiaxial experiments presented in Fig. 6 lie one atop the other provide experimental evidence that the mechanical response for the pleural membrane is isotropic. This agrees with the experimental findings of Humphrey et al. (1987), which were performed on a canine sample. What is immediately apparent when comparing our Fig. 6 and their Fig. 1 is the large animal-to-animal and species-to-species variability that exists in the limiting state of equibiaxial stretch, i.e., $\eta_{\max} = 1/\alpha$, which spans a range of approximately $\lambda_{\max} \in [1.3, 1.7]$ in just these three data sets, thereby implicating a large variability in α of at least $\alpha \in [0.9, 1.8]$. This is true of other tissues, too, e.g., large variability in the limiting state of stretch is observed in bioprosthetic heart-valve tissues reported on by Vesely (1996). The curves present in Fig. 6 represent optimal fits of Eq. (21) to the experimental data, where E was forced to be the same between them.

The idea of constructing an elastic theory for isotropic membranes using the smaller set of invariants defined over $\mathbb{R}^2$, as considered in the early literature on the topic and adopted herein, instead of using a larger set of invariants defined over $\mathbb{R}^3$, but specialized for membranes, appears to have originated in a paper by Wong and Shield (1969). Holzapfel and Ogden (2009) have argued the point that "there is a significant difference between the planar specialization of a three-dimensional strain-energy function and an *a priori* two-dimensional strain-energy function." This is especially true for anisotropic materials. In this paper, however, the material of study exhibits mechanical isotropy, even though it has a heterogeneous fibrous microstructure.

We agree with their statement "in a two-dimensional theory a significant part of the three-dimensional constitutive law is missing." In fact, invariants that generate stress and strain induced anisotropies, specifically $\bar{\mathsf{S}} \odot \mathsf{C}^{-1} + \mathsf{C}^{-1} \odot \bar{\mathsf{S}}$ and $\bar{\mathsf{E}}_G \odot \mathsf{C} + \mathsf{C} \cdot \bar{\mathsf{E}}_G$ in our three-dimensional model (Freed and Einstein 2013), do not arise here in our two-dimensional model. Therein, C and C^{-1} are the deformation metrics of Green and Cauchy, S is the second Piola-Kirchhoff stress and E_G denotes Green strain.

The Fungean model in this paper remains mechanically isotropic, with the classic *elastic constants* of a Hookean solid becoming *elastic tangent functions* in this model. This is because the tensorial dependence of $\mathbb{M}$ is in terms of the two, isotropic, outer and inner, tensor products $\mathsf{U}^{-1} \otimes \mathsf{U}^{-1}$ and $\mathsf{U}^{-1} \odot \mathsf{U}^{-1}$ present in the isotropic Hookean model. The stress and strain induced anisotropies mentioned above, as direct consequences of implicit elastic theory, are topics left for future research, both experimentally and theoretically. To construct a Fungean material model for an anisotropic membrane, one would need to consider either those planar-specialized invariants that exist over $\mathbb{R}^3$, which have been neglected in our construction over $\mathbb{R}^2$, or introduce, as a minimum, a third tensor into the invariant construction process, e.g., the case of transverse isotropy (cf. Spencer 1971, 1972).

The strain-limiting terms $1/(1 - \alpha\eta)$ and $1/(1 - 2\beta I_3)$ present in $\tilde{E}$ and $\tilde{\mu}$ of our Fungean model are what Freed and Einstein (2013) refer to as *Rajagopal effects*. The exponential effect of $E + \alpha\tau$ is what they refer to as a *Fung effect*.

There is no parameter in our Fungean model that provides information regarding how the behavior is to transition from a Hookean response into a saturated Carton response. In other words, there is nothing in this model that allows for adjustment in the shape of the heel region in a typical uni-

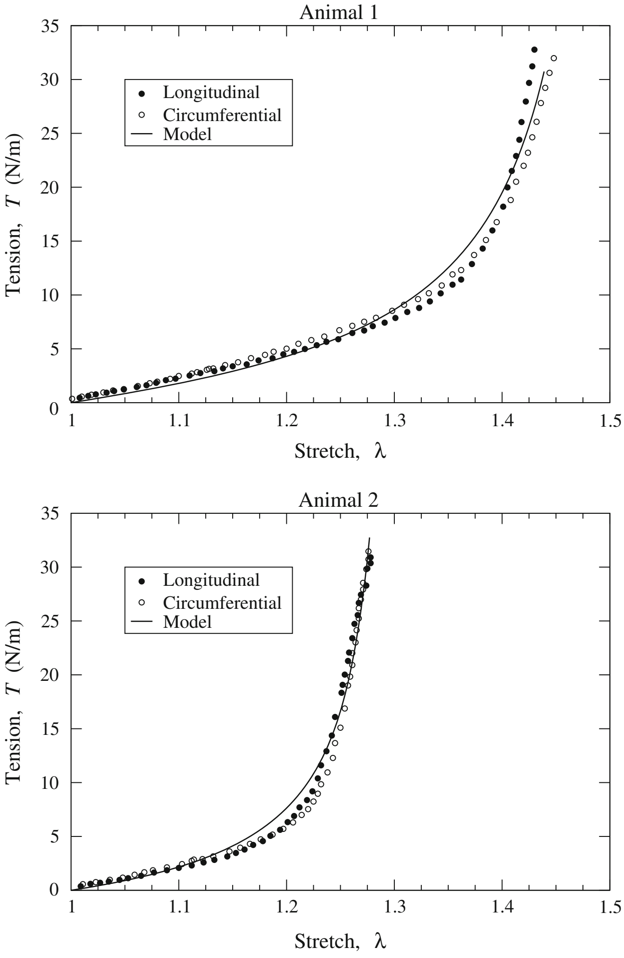

Fig. 6 Longitudinal and transverse responses of two porcine pleura subjected to equibiaxial extensions. For animal 1, the displayed curve is described by parameters $\alpha = 1.2$ and $E = 8$ N/m. For animal 2, it is described by parameters $\alpha = 1.8$ and $E = 8$ N/m. The elastic modulus E was selected to be the same for both animals, only permitting α to vary between them. Parameters μ and β are insensitive to equibiaxial extension

axial experiment done on tissue. If our simple model fails to describe data from a data set of interest to you over this region of transition, then these data that lie in the in-between heel region will likely confuse your efforts to employ an automated optimization algorithm.

Although we advise against this temptation, if one has a need to model the heel region of a stretch/tension curve with greater fidelity than Eqs. (18, 19) permit, one might want to consider

$$\mathfrak{W}_i(\eta, \tau) = \frac{\tau^2}{2E}(1 - \alpha\eta)^m \quad \&$$

$$\mathfrak{W}_d(\bar{\mathsf{E}}, \bar{\mathsf{T}}) = \frac{I_2}{2\mu}(1 - \beta I_3)^n$$

where m and n would modulate the shape of the heel region, with $m = n = 1$ reducing the above model to our original model (18, 19).

6 Summary

An elastic theory for membranes has been derived and applied to porcine pleura. The theory is novel in that it is based upon an additive dilatoric/deviatoric split in a differential change of deformation, in particular, of the right stretch tensor. A consequence of this hypothesis is that the elastic energy stored internally by the material, caused by external work being done on it, splits into independent isotropic and deviatoric contributions—a conjecture that is supported by the experimental data presented in this paper.

The theory is also novel in that its mathematical structure, derived from thermodynamics, is implicit; its internal energy function can depend upon both stress and strain. For specific choices in its two energy functions, a Fungean membrane has been shown to exist whose tangent moduli have the same tensorial structure as those of a Hookean membrane, with its elastic constants being replaced by elastic tangent functions. The resulting constitutive formulæ produce exponential-like responses indicative of behaviors exhibited by soft biological tissues whose extracellular constructions consist of collagen fibers.

A sequence of proportional biaxial experiments done on a visceral pleural membrane validate the model. These experiments found the membrane to be mechanically isotropic, and the mechanical work done on it to be separable into independent isotropic and deviatoric contributions.

This technique of splitting the work into isotropic and deviatoric constituents greatly facilitates parameter estimation in that there are just two parameters for each energy contribution. One for the small-deformation Hookean response. The other for the large-deformation asymptotic response. Where the model may be wanting is that it has no parameters to adjust for the material response that lies between these two limiting states. By splitting the data into isotropic and deviatoric responses, as we have done in Figs. 1 and 2, the two facets of our model can be parameterized in an uncoupled approach that maximizes parameter sensitivity to the data. A separate issue has to do with parameter variability. A standard deviation in error between data sets determined from multiple tissue samples will be much greater for the Fungean parameters α and β than for the Hookean parameters E and μ. How much and what its impact would be on material modeling in applications has not yet been studied.

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